Optimal Delay in Networks with Controlled Mobility

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***Abstract*— We consider a network in which a set of vehicles is responsible for the pickup and delivery of messages that arrive according to Poisson process with message pickup and delivery locations distributed uniformly at random in a region of bounded area A. The vehicles are required to pickup and deliver the messages so that the average delay is minimized.**

In this paper, we provide lower bounds on the delay achievable by fully controlled policies, depending on the information constraint in place. We prove that for any policies in which only the source location information is known upon message arrival, the optimal average delay scaling is $\Theta(\lambda(n)A/v^2n)$. If in addition to source location, destination locations of messages are known to the vehicles, the optimal average delay scaling can be reduced to $\Theta(\lambda(n)A/v^2n^{3/2})$. We note that these scaling bounds are achievable given the service policies we have previously described in [1].

I. Introduction

We first present a model for a message passing network, and describe the Dynamic Pickup-Delivery Problem (DPDP) for serving messages in this network. We will then motivate this problem in the context of vehicle routing.

A. Model

1) Vehicles and Messages: Let there be n vehicles in a geographic area $\mathcal{A} \subset \mathbb{R}^2$, which is a convex, compact set with volume A. For simplicity, we consider $\mathcal{A} = [0, \sqrt{A}]^2$. Each vehicle may move in any direction at any time with a velocity of magnitude $\leq v$. For simplicity, assume $v = 1$.

Messages are generated according to a Poisson process with time intensity $\lambda(n)$. We will examine the heavy traffic case, that is, we assume that the rate of messages arriving to the system increases much faster than the number of vehicles in the system available to service these messages, equivalently $\lambda = \Omega(n)$[1]. The precise required scaling of $\lambda(n)$ will be stated in the theorems. [2]

Associated with each message j are source and destination locations denoted by $s(j) \in \mathcal{A}$ and $d(j) \in \mathcal{A}$ respectively. Both the source and destination locations are independently and identically distributed (IID) uniformly in $\mathcal{A}$.

The messages need to be picked up from their source locations and delivered to their destination locations by the vehicles. A message is picked up (delivered) when a vehicle spends a fixed on-site service time of $\bar{s}(n)$ at the source (delivery) location to pick up (deliver) the message. For stability reasons, the service time, $\bar{s}(n)$, is implictly bounded by a function of n and $\lambda(n)$. To see this, compare the system to an M/D/n queue with service times defined to be only the total time spent in onsite service, that is, the total service time per message is $2\bar{s}(n)$. The average system utilization for this system is $\rho = \frac{\lambda(n)2\bar{s}(n)}{n}$, the product of the arrival rate and the service time per message divided by the number of vehicles that serve these messages. The stability condition for this system is $\rho < 1 \Leftrightarrow \bar{s}(n) < \frac{n}{2\lambda(n)}$. Therefore, the maximum onsite service time $\bar{s}(n)$ supportable by a stable system is implicitly a function of n and $\lambda(n)$.

We make the requirement that the vehicle that picks up a message must be the one that delivers it. That is, messages may not be transferred between vehicles after they have been picked up. We assume that each vehicle can carry an unlimited number of messages at any time.

2) Control Policies: A control policy, π, is a set of decision making rules for each vehicle that decides the pickup and delivery schedule of arriving messages, based on a set of constraints on the information available to the vehicle. Vehicles then follow predetermined protocols to service these messages without any real-time communication between vehicles to coordinate their protocols dynamically.

We will restrict our attention to time invariant policies that

[1]Recall the following notation: $(i) f(n) = O(g(n))$ means that $\exists$ a constant c and integer N such that $f(n) \leq cg(n), \forall n \geq N$. (ii) $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$. (iii) $f(n) = \Theta(g(n))$ means that $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

[2]For ease of notation, $\lambda(n)$ will usually be written as just λ.

satisfy the following spatial properties. For each vehicle i, the control policy induces a scaled probability density p_i on $\mathcal{A} \times \mathcal{A}$, such that vehicle i serves messages with source-destination location pairs in a subset $S \subset \mathcal{A} \times \mathcal{A}$ at rate $\lambda_i(S)$, where

$$\lambda_i(S) = \lambda \int_S p_i(x, y) dx dy.$$

Furthermore, we also require that the assignment of messages is independent given these densities. With this, the set $\{p_i(x, y)\}_{i=1}^n$ may then be used to fully describe the assignment policy as follows. If $p_i(x, y)$ is nonzero for exactly one i for some fixed pair (x, y), then every message that arrives to the system at location x that is destined for y will be assigned to vehicle i. If multiple vehicles have nonzero $p_i(x, y)$ at the pair (x, y), then the message is assigned independently at random to one of these vehicles. This assignment may be achieved by a centralized controller or with limited communication between the affected vehicles. The probability of each assignment is weighted by the magnitude of the nonzero density at $p_i(x, y)$. This policy restriction allows messages to be assigned to vehicles unambiguously based on message location, reducing the need for real-time communication between vehicles regarding message assignments. Note that a policy may be specified by a set of rules that induce a set of densities $\{p_i(x, y)\}_{i=1}^n$, or this set of densities may be specified directly.

For a fixed policy π, let $\lambda_i(\pi) = \lambda \int_A p_i(x, y) dx dy$ denote the total rate of messages served by vehicle i where $p_i(x, y)$ is the density induced by the policy π.

In deciding which messages to pickup and in what order to serve them, vehicles may only use information about the arriving messages according to the information structure in place. We assume that the vehicles have no knowledge of individual messages before they arrive although the overall message arrival process and source and destination distributions are known. In particular, we consider two types of information structure:

(a) *Source only information*: When a message arrives, its source location is known by all vehicles within some region of interest, but vehicles do not know the destination of messages until they pick them up; and

(b) *Source-destination information*: When a message arrives, both its source and also its destination location are known by all vehicles within some region of interest.

Let Π_{SO} (respectively Π_{SD}) denote the set of all time invariant and spatially based policies satisfying the properties above and using only information available in the Source Only (respectively Source and Destination) information structure. Our results will show that the performance of optimal control policies is significantly affected by the particular information structure in place.

3) Performance Metrics: The delay of message j, denoted $W(j)$, is defined to be the elapsed time between the message's arrival to the system and its delivery to its destination location. This includes any time the message waits to be picked up, the onsite service time for pickup, travel time on the vehicle before arriving at the delivery location, and finally onsite service time for delivery. The quantity W is defined to be

$$W = \limsup_{j \to \infty} \mathbb{E}[W(j)]. \tag{1}$$

If the limit exists when limsup is replaced by limit in the definition above, W is the steady state expected value of the delay of a message in the system. We say that the system is stable if $W < \infty$. A necessary condition for the existence of a stable policy is $\rho = \frac{\lambda(n) 2\bar{s}(n)}{n} < 1$. If π is a stable policy, $W(\pi)$ is defined to be the average delay associated with that policy.

We may also define the average waiting time experienced by messages served by a particular vehicle:

$$W_i = \limsup_{j \to \infty} \mathbb{E}[W(j) \cdot 1_{\text{message } j \text{ served by vehicle } i}] \tag{2}$$

where 1_A is the indicator variable of the event A. The system is then stable if $\sum_{i=1}^n \lambda_i = \lambda$ and $W_i < \infty, \forall i$. For a single vehicle with message arrival rate λ_i, a necessary condition for the existence of a stable policy is $\rho_i = \lambda_i 2\bar{s}(n) < 1$. Therefore, $\rho_i < 1, \forall i$ is necessary to guarantee $W_i < \infty, \forall i$. If π is a stable policy, $W_i(\pi)$ is defined to the be the average delay of messages served by vehicle i under that policy. The average waiting time of all messages in the system $W(\pi)$ above is then equivalent to $W(\pi) = \sum_{i=1}^n \frac{\lambda_i(\pi)}{\lambda} W_i(\pi)$.

4) Problem Definition: We seek stable policies that minimize the average delay per message. We restrict our attention to time invariant and spatially based policies and utilize information only according to the particular information structure in place. That is, we must solve the following optimization problem which will be denoted as $\mathcal{OPT}_1$:

$$\min_{\pi \in \Pi_I} \qquad W = \sum_{i=1}^n \frac{\lambda_i(\pi)}{\lambda} W_i(\pi) \tag{3}$$

$$s.t. \qquad \sum_{i=1}^n \lambda_i(\pi) = \lambda \tag{4}$$

$$\rho_i(\pi) < 1, \forall i \tag{5}$$

Note that each policy π under consideration may be uniquely described by the set of densities $\{p_i(x,y)\}_{i=1}^n$ and that each of the $\lambda_i(\pi)$ is defined in terms of $p_i(x,y)$ as $\lambda_i(\pi) = \lambda \int_{\mathcal{A}} \int_{\mathcal{A}} p_i(x,y)dxdy$. Therefore, the optimization above is implicitly a function of $p_i(x,y)$ alone.

We call this problem the Dynamic Pickup-Delivery Problem (DPDP).

Our model and problem thus stated, we make the connection to various problems in the field of vehicle routing.

B. Relation to Previous Work

In the DPDP, information about a particular demand is not available until the demand has arrived. Because the information available to the controllers evolves according to the arrival of new demands, the DPDP problem is naturally related to the problem of dynamic vehicle routing. In constrast to classical static vehicle routing problems, the solution to this dynamic problem takes the form not of a fixed route through a given set of demands, but is instead a control policy that determines how the routes evolve in time as a function of the demands in the system. Dynamic vehicle routing problems have received much less attention than their static counterparts, but recent surveys on dynamic vehicle routing problems include [2] and [3].

Because each message in the DPDP must be both picked up and delivered, this problem may be further categorized as a Pickup and Delivery problem (PDP). Recent surveys of pickup and delivery problems include [4] and [5], although most of these works focus on static pickup and delivery problems only. If demands are interpreted to be people to be picked up and transported from one location to another, this is also known as the Dial-a-Ride problem (DARP). Other applications of PDPs include courier services and less than truckload trucking.

There are significant theoretical results on a simplified dynamic VR problem, the Dynamic Traveling Repairperson Problem (DTRP), proposed by Bertsimas and van Ryzin [6]–[8]. In the DTRP, demand service requires only a single onsite service time, without the delivery requirement of the DPDP studied here. Intuitively, our problem seems very similar to the DTRP as both the pickup and delivery of messages in DPDP can be treated as separate requests in the DTRP problem setup. However, the pick-up and delivery services of a single message are strongly linked, making our problem significantly distinct from the DTRP. Because the DTRP treats a simplified version of our problem, the lower bounds on DTRP delay presented will serve as a lower bound over a DPDP policies. The DTRP will be discussed in more detail in Section II-B.

The Dynamic Pick-up Deliveryperson Problem (DPDP) has been previously studied in [9]. This work provides a theoretical analysis of a simplified pickup and delivery problem, considering only the single vehicle case, and focusing mainly on vehicles with unit capacity. We are interested in the multi-vehicle case with infinite capacity vehicles, and so our work here is a significant extension of the DPDP analysis in [9].

C. Main Results

The goal of the current paper is to find the minimum average message delay achievable by any valid control policy that satisfies the assumptions in Section I-A.2. We further divide the control policies into two categories based on the information structure in place for making the control decisions. In the Source Only structure, only message source locations are known before the message is picked up. In the Source and Destination structure, both the source and destination locations of messages are known as soon as the message arrives. We will prove lower bounds on the average message delays achieveable by control policies from these two groups. We will note that the delay bounds derived here in fact match the order of the delay scaling demonstrated by the policies in [1], and therefore these policies are order optimal and the lower bound may be achieved. In particular, we prove the following two theorems:

Theorem 1:

(a) For any policy in Π_{SO} under the Source Only information structure, the average delay per message scales as

$$W_{SO} \geq \gamma^2 \left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n} \right) - \frac{n(1-2\rho)}{2\lambda(n)} \tag{6}$$

with constant $\gamma = 2/3\sqrt{2\pi}$. For $\lambda(n) = \Omega(n)$ this bound scales as $W_{SO} = \Omega(\lambda(n)A/v^2(1-\rho)^2 n)$.

(b) *(Theorem 2 from [1])* Further, there exists a policy using Source Only information, for which the average delay scales as $O(\lambda(n)A/v^2(1-\rho)^2 n)$ for all $\lambda(n)$. Therefore the lower bound scaling is achievable.

Theorem 2:

(a) For any policy in Π_{SD} under the Source-Destination information structure, the average delay per message with $\lambda(n) = \Omega(n)$ scales as

$$W_{SD} \geq \frac{\gamma^2}{4} \frac{\lambda(n)A}{v^2(1-\rho)^2 n^{3/2}} \tag{7}$$

with constant $\gamma = 2/3\sqrt{2\pi}$.

(b) *(Theorem 3 from [1])* Further, there exists a policy using Source and Destination information for which the delay scales as $O(\lambda(n)A/v^2(1-\rho)^2 n^{3/2})$ for $\lambda(n) = \Omega(n^{3/2})$. Therefore the lower bound scaling is achievable in heavy traffic.

Theorem 1 first quantifies the achievable performance for control policies with some minimal amount of information. Theorem 2 then quantifies the effect of additional information on achievable performance. We note that even the full information case is greater than the results on the DTRP in [7] by a factor of $\sqrt{n}$.

D. Organization

The rest of the paper is organized as follows. In Section II, we provide some additional notation and detail some mathematical results from vehicle routing that will be useful in our analysis. In Section III, we give the proofs of the lower bounds in Theorem 1(a) and Theorem 2(a). Section IV describes and analyzes policies that achieves the claimed performance. Finally, in Section V we present discussion and directions for future work.

II. Preliminary Technical Results

Before proving Theorems 1(a) and 2(a), we first provide some notation and a more detailed discussion of the results on the Dynamic Traveling Repairperson Problem from [6]–[8].

First, we will adopt the following notation: for any reasonable function $g(\cdot)$

$$\mathbb{E}_\theta[g(\cdot)] \triangleq \int_{\mathcal{A}} g(\theta)\ d\theta.$$

Essentially, $\mathbb{E}_\theta$ is the standard (Lebesgue) integration. We retain the reference to the variable θ in order to sometimes differentiate the integration with respect to source and destination location.

A. Induced Service Densities

In Section I-A.2, we assumed the existence of a scaled service density $p_i(x, y)$ for each vehicle that gives the steady state probability of the vehicle i serving a given message that originates at x and is destined for y. In this section, we provide some additional properties of the $p_i(x, y)$ that will be useful in the optimization of $\mathcal{OPT}_1$.

Unless specified otherwise, in the rest of the paper, variables x and y will be used to refer to source and destination locations respectively. By definition,

$$p_i(x, y) \geq 0,\ \forall(x, y).$$

For stability, the n vehicles must collectively serve all the arriving traffic. That is,

$$\sum_{i=1}^{n} p_i(x, y) \quad = \quad \phi_s(x)\phi_d(y) \ = \ \frac{1}{A^2}. \tag{8}$$

Further, from the definition of λ_i we have,

$$\mathbb{E}_x[\mathbb{E}_y[p_i(x, y)]] \quad = \quad \frac{\lambda_i}{\lambda}. \tag{9}$$

Finally, note that since the vehicle must both pickup and deliver each message assigned to it, exactly half of the service locations visited by a vehicle are pickup locations. Therefore, we may define $f_i(\zeta), \zeta \in \mathcal{A}$, the normalized probability density of vehicle i servicing (i.e. either picking up or delivering) a message at location ζ, as a uniform mixture of the pickup and delivery distributions. That is,

$$f_i(\zeta) \quad = \quad \frac{1}{2}\frac{\lambda}{\lambda_i}\left[\mathbb{E}_x[p_i(\cdot, \zeta)] + \mathbb{E}_y[p_i(\zeta, \cdot)]\right]. \tag{10}$$

B. The Dynamic Traveling Repairperson Problem

Before beginning our analysis of the DPDP problem, it is important to more precisely state a few results on the related Dynamic Traveling Repair-person Problem (DTRP) that were proven by Bertsimas and van Ryzin [6]–[8] and that will be used in our lower bound analysis of the DPDP. The DTRP considers the case in which demands arrive to a convex environment $\mathcal{A}$ of area A according to some arrival process with demands being randomly located in the region according to some distribution. A demand is serviced when a vehicle arrives to the demand location and spends a random amount of onsite service time, s, to service the demand. To perform these services, there are n vehicles that travel with bounded velocity $\leq v$ within $\mathcal{A}$. The average system utilization is defined in the standard queueing theory sense to be $\rho = \lambda\bar{s}/n$. The demands are to be serviced in such a way that all demands are eventually serviced and average delay between arrival and service of the demands, W, is minimized.

In the case that demands arrive according to a Poisson process with rate λ and demand locations are independently and identically uniformly distributed in $\mathcal{A}$, the average delay of message in the system is:

Theorem 3: (Theorem 2 in [7])

$$W \geq \gamma^2 \frac{\lambda A}{v^2(1-\rho)^2 n^2} - \frac{1-2\rho}{2\lambda} \tag{11}$$

for constant $\gamma = 2/3\sqrt{2\pi}$.

The more general case of non-Poisson arrivals and nonuniform iid demand distributions is treated in [8]. Although the DPDP presently considers Poisson arrivals, [8] shows that the Poisson assumption is easily taken care of with little change to the delay results. Further, they consider two classes of policies: spatially unbiased and spatially biased. Spatially unbiased policies require that the average expected delay of a message is the same regardless of the demand location, and spatially biased policies simply remove this restriction. Therefore, if we do not care about the notion of spatial biasedness, the results on spatially biased policies provide the strongest result. Below we state a slightly modified version of result in [8] on the average delay over all messages that arrive according to demand distribution $f(\zeta)$ and are served under a spatially biased policy.

Theorem 4: (Theorem 2 from [8] (modified))

With $\lambda = \Omega(n)$, the average delay W scales as

$$W \geq 2\gamma^2 \frac{\lambda (E_\zeta[f^{2/3}])^3}{v^2(1-\rho)^2 n^2} \tag{12}$$

where $\gamma \geq \frac{2}{3\sqrt{2\pi}}$.

Theorem 4 follows as stated above with a slight modification of the proof in [8]. The modified proof may be found in the appendix.

In the following section, we construct DTRP queues to lower bound the performance of our actual DPDP queues and apply the above theorems to analyze these lower bounds.

III. Lower Bounds

In this section we prove the claimed lower bounds of Theorems 1(a) and 2(a) for arbitrary policies.

A. Lower bound: Source only

We consider the Source Only information structure to be one of minimal information. Because destination locations are not known immediatly upon message arrival, this information may not be exploited when assigning messages to vehicles.

Theorem 1(a): For any policy in Π_{SO} under the Source Only information structure, the average delay per message scales as

$$W_{SO} \geq \gamma^2 \left(\frac{\lambda(n) A}{v^2(1-\rho)^2 n} \right) - \frac{n(1-2\rho)}{2\lambda(n)} \tag{13}$$

with constant $\gamma = 2/3\sqrt{2\pi}$. For $\lambda(n) = \Omega(n)$ this bound scales as $W_{SO} = \Omega(\lambda(n)A/v^2(1-\rho)^2 n)$.

Proof: Consider a fixed stable service policy in Π_{SO}. Each message is assigned to its vehicle immediately upon arrival. Each vehicle can then be treated as a queue of messages that have been assigned to it. Consider the queue at vehicle i. Because a spatially based assignment process is used and the locations of successive messages are independent of each other, the assignment of messages to vehicles induces a splitting of the overall Poisson arrival process. The arrival process to vehicle i is a Poisson process of rate λ_i.

To lower bound the average delay of messages, we consider a simplified system in which the same message assignment process holds, but messages arrive directly at the vehicle according to a Poisson process of rate λ_i. That is, vehicles do not spend any time in picking up messages. This simplified system naturally has lower delay than the original system. Because vehicles only have access to information about the source locations of messages, the destination locations of the messages may not be exploited by the message assignment policy. Since the distribution of destination locations is independent of the arrival locations, the distribution of the destination locations of the messages assigned to a single vehicle is the same as that of the overall destination process, irrespective of policy. So for any policy in Π_{SO}, each vehicle will service messages with destination locations distributed uniformly at random in $\mathcal{A}$. Thus, it is sufficient to lower bound delay of the following simplified system: each vehicle has messages arriving according to a Poisson process of rate λ_i with uniformly distributed delivery locations.

For each vehicle, this delivery problem may be formulated as a Dynamic Traveling Repairperson Problem in which a single vehicle is responsible for servicing all messages that arrive with rate λ_i, uniformly distributed in $\mathcal{A}$. In this formulation, $\rho_i = 2\lambda_i \bar{s}(n)$, where the factor 2 reflects the two onsite service times required for pickup and delivery. Therefore, applying the DTRP results of Theorem 3 to this formulation, we obtain the average delay for messages served by a single vehicle with message arrival rate λ_i:

$$W_i \geq \gamma^2 \left(\frac{\lambda_i A}{v^2(1-\rho_i)^2} \right) - \frac{1-2\rho_i}{2\lambda_i} \tag{14}$$

To bound the average delay over all messages, we must solve the following optimization as in $\mathcal{OPT}_1$:

$$\min_{\{\lambda_i\}_{i=1}^n} \quad \sum_{i=1}^n \frac{\lambda_i}{\lambda} \left(\gamma^2 \left(\frac{\lambda_i A}{v^2(1-\rho_i)^2} \right) - \frac{1-2\rho_i}{2\lambda_i} \right) \tag{15}$$

$$s.t. \quad \sum_{i=1}^n \lambda_i = \lambda \tag{16}$$

For the moment, we assume that $\bar{s}(n)$ is sufficiently small such that $\rho_i < 1, \forall i$ and constraint (5) is satisfied.

Removing constant terms and noting that $\sum_{i=1}^n \rho_i =$

$\sum_{i=1}^{n} 2\lambda_i \bar{s}(n) = n\rho$, this is equivalent to:

$$\min_{\{\lambda_i\}_{i=1}^n} \quad \sum_{i=1}^{n} \frac{\lambda_i^2}{(1-\rho_i)^2} = \frac{\lambda_i^2}{(1-\lambda_i \bar{s})^2} \tag{17}$$

$$s.t. \quad \sum_{i=1}^{n} \lambda_i = \lambda \tag{18}$$

This optimization is straightforward to solve using Lagrange multipliers. We find that the optimal solution is $\lambda_i = \frac{\lambda(n)}{n}, \forall i$. The stability constraint (5) is then satisfied for $\bar{s}(n) < n/2\lambda(n)$. Therefore, we have the following lower bound on the average delay over all vehicles:

$$W_{SO} \geq \gamma^2 \left(\frac{\lambda A}{v^2(1-\rho)^2 n} \right) - \frac{n(1-2\rho)}{2\lambda} \tag{19}$$

This lower bound holds for all $\lambda(n)$ scalings. If $\lambda(n) = \Omega(n)$, however, the first term in (19) dominates the second and we have therefore proven Theorem 1(a):

$$W_{SO} = \Omega \left(\frac{\lambda A}{v^2(1-\rho)^2 n} \right) \tag{20}$$

■

B. Lower bound: Source-destination

If both the Source and Destination locations are known upon message arrival, assignment policies may exploit this information to limit the area covered by each vehicle in making its pickups and deliveries. We show that this has the effect of reducing the minimum average delay of messages in the system.

Theorem 2(a): For any policy in Π_{SD} under the Source-Destination information structure, the average delay per message with $\lambda(n) = \Omega(n)$ scales as

$$W_{SD} \geq \frac{\gamma^2}{4} \frac{\lambda A}{v^2(1-\rho)^2 n^{3/2}} \tag{21}$$

with constant $\gamma = 2/3\sqrt{2\pi}$.

Proof: Now consider a fixed stable service policy in Π_{SD} based on source-destination information. Given the source and destination assignment distributions induced by this policy, we will again construct a simplified DTRP system, the delay of which will lower bound the delay encountered by messages in the original DPDP system.

As above, examine the service policy of a single vehicle i and consider the queue induced by messages that are assigned to this vehicle. The DTRP demand location associated with each message is selected uniformly at random between the source $s(j)$ or the destination $d(j)$ location of the message. That is, instead of performing both pickup and delivery as in the DPDP or delivery only as in the proof of the Source Only lower bound above, this DTRP visits either the pickup location or the delivery location of a single message, with either location being chosen with probability 1/2. Therefore, the distribution of demand locations arriving to this DTRP queue is the same as the normalized density of vehicle i's pickup and delivery locations, i.e. $f_i(\zeta)$. Note that since the DTRP queue ignores either the pickup or delivery requirement of each message, the delay of the demands in the DTRP queue is less than that of messages in the original system.

This DTRP queue fits the framework of the single vehicle Dynamic Traveling Repairperson Problem with generalized demand distributions. Then, according to Theorem 4, we have the following bound on minimum delay for a single vehicle policy with demand distribution $f_i(\zeta)$, arrival rate λ_i, and $\rho = \lambda_i \bar{s}$:

$$W_i \geq \gamma^2 \frac{\lambda_i (E_\zeta[f_i^{2/3}])^3}{(1-\rho_i)^2} \tag{22}$$

Recall the relation of $f_i(\zeta)$ to $p_i(x,y)$ and the constraints imposed by the delivery requirement on $p_i(x,y)$ from equations (8), (9) and (10). From (10):

$$f_i(\zeta) = \frac{1}{2} \frac{\lambda}{\lambda_i} \left[\mathbb{E}_x[p_i(x,\zeta)] + \mathbb{E}_y[p_i(\zeta,y)] \right].$$

This implies the following two lower bounds:

$$\mathbb{E}[f_i^{2/3}] \quad \geq \quad \left(\frac{\lambda}{2\lambda_i} \right)^{2/3} \mathbb{E}_\zeta[\mathbb{E}_x[p_i(x,\zeta)]^{2/3}] \tag{23}$$

$$\geq \quad \left(\frac{\lambda}{2\lambda_i} \right)^{2/3} \mathbb{E}_\zeta[\mathbb{E}_y[p_i(\zeta,y)]^{2/3}]. \tag{24}$$

Now, $p_i(x,y)$ have the following basic constraints (implied from (8) and (9)):

$$p_i(x,y) \quad \in \quad \left[0, \frac{1}{A^2} \right], \tag{25}$$

$$\mathbb{E}_x[\mathbb{E}_y[p_i]] \quad = \quad \frac{\lambda_i}{\lambda}. \tag{26}$$

We may combine the two lower bounds above to form the following optimization problem $\mathcal{OPT}_2$ which will then be used to lower bound the delay of a single vehicle policy with fixed arrival rate λ_i:

$$\begin{aligned} \min_{p_i(x,y)} \quad & \frac{1}{2}(\mathbb{E}_\zeta[(\mathbb{E}_x[p_i(x,\zeta)])^{2/3}] + \mathbb{E}_\zeta[(\mathbb{E}_y[p_i(\zeta,y)])^{2/3}]) \\ \text{subject to} \quad & p_i(x,y) \in [0, 1/A^2], \\ & \mathbb{E}_x[\mathbb{E}_y[p_i(x,y)]] = \mathbb{E}_y[\mathbb{E}_x[p_i(x,y)]] = \frac{\lambda_i}{\lambda}. \end{aligned}$$

Consider a convex combination of two densities satisfying (25) and (26), i.e. $p_i^3(x,y) = \alpha p_i^1(x,y) + (1-\alpha)p_i^2(x,y), \forall x,y \in \mathcal{A}$. It is easy to see that the set of valid probability distributions satisfying (25) and (26) is convex.

Further, by the concavity of $(\cdot)^{2/3}$,

$$\begin{aligned}
(\mathbb{E}_x[p_i^3(x,\zeta)])^{2/3} &= (\alpha\mathbb{E}_x[p_i^1(x,\zeta)] + (1-\alpha)\mathbb{E}_x[p_i^2(x,\zeta)])^{2/3} \\
&\geq \alpha(\mathbb{E}_x[p_i^1(x,\zeta)])^{2/3} + \\
&\qquad (1-\alpha)(\mathbb{E}_x[p_i^2(x,\zeta)])^{2/3} \\
\mathbb{E}_\zeta[(\mathbb{E}_x[p_i^3(x,\zeta)])^{2/3}] &\geq \alpha\mathbb{E}_\zeta[(\mathbb{E}_x[p_i^1(x,\zeta)])^{2/3}] + \\
&\qquad (1-\alpha)\mathbb{E}_\zeta[(\mathbb{E}_x[p_i^2(x,\zeta)])^{2/3}]
\end{aligned}$$

Therefore both of the lower bounds (23) and (24) are concave in $p_i(x,y)$ and so is their sum. Thus, $\mathcal{OPT}_2$ is a concave minimization over a convex set. Hence, it must attain its optima on the boundary of the feasible bounded convex set.

The boundary of the constraint set defined by (25)-(26) implies that $p_i(x,y) \in \{0, 1/A^2\}$ for all (x,y) (almost surely w.r.t. Lebesgue measure). Condition (26), along with this implication, will provide the following complete characterization of boundary.

$$p_i(x,y) = \begin{cases} \frac{1}{A^2} & \text{for all } x \in \mathcal{A}_1, y \in \mathcal{A}_2 \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

with $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ of areas such that $A_1 A_2 = \frac{A^2\lambda_i}{\lambda}$.

To minimize the cost function, we must select the boundary points where the areas of A_1 and A_2 are equal, i.e. both are equal to $A\sqrt{\frac{\lambda_i}{\lambda}}$.

For any p_i satisfying the above properties we have:

$$\mathbb{E}_x[p_i(x,\zeta)] = A\sqrt{\frac{\lambda_i}{\lambda}}\frac{1}{A^2} = \frac{1}{A}\sqrt{\frac{\lambda_i}{\lambda}}$$

and

$$\mathbb{E}_\zeta[\mathbb{E}_x[p_i(x,\zeta)]^{2/3}] = A\sqrt{\frac{\lambda_i}{\lambda}}\left(\frac{1}{A}\sqrt{\frac{\lambda_i}{\lambda}}\right)^{2/3} = A^{1/3}\left(\frac{\lambda_i}{\lambda}\right)^{5/6}$$

and therefore the bound (23) on $\mathbb{E}[f_i^{2/3}]$ becomes

$$\mathbb{E}[f_i^{2/3}] \geq \left(\frac{\lambda_i}{2\lambda}\right)^{2/3}\mathbb{E}_\zeta[\mathbb{E}_x[p_i(x,\zeta)]^{2/3}] = \frac{1}{2^{2/3}}A^{1/3}\left(\frac{\lambda_i}{\lambda}\right)^{1/6} \tag{28}$$

Cubing and then substituting this bound into equation (22), we thus have the following bound on minimum delay for messages served by vehicle i:

$$W_i \geq \frac{\gamma^2}{4}\frac{\lambda_i A}{v^2(1-\rho_i)^2}\sqrt{\frac{\lambda_i}{\lambda}} \tag{29}$$

With this delay solved for a single vehicle serving messages at rate λ_i, we again formulate the optimation problem over all vehicles which reduces similar to the above, to:

$$\min_{\{\lambda_i\}_{i=1}^n} \quad \frac{\lambda_i^{5/2}}{(1-\lambda_i\bar{s})^2} \tag{30}$$

$$s.t. \quad \sum_{i=1}^n \lambda_i = \lambda \tag{31}$$

As above, this average delay is minimized with all λ_i equal to λ/n. Therefore, we have the following lower bound on the average delay with Source and Destination information:

$$W_{SD} \geq \frac{\gamma^2}{4}\frac{\lambda(n)A}{v^2(1-\rho)^2 n^{3/2}} \tag{32}$$

Since $\lambda = \Omega(n)$ was required for the application of the DTRP theorem with generalized demand distributions, this bound is valid for $\lambda = \Omega(n)$ and Theorem 2(a) is proven. ■

IV. Policies

In this section, we briefly describe policies that achieve the delay performance claimed in Theorems 1(b) and 2(b) for Source Only and Source Destination information respectively. The delay analysis of these policies may be found in [1].

A. Source Only Policy

Recall that in the Source Only information structure, vehicles do not know the destination of messages before they are picked up, thus this information may not be used by vehicles in deciding which messages to pick up.

In the source only policy described below, each message is assigned to a random vehicle. Messages are not served immediately upon arrival but all arrivals occurring in a interval of length T are accumulated into a batch. Given the fixed collection of pickup locations in the batch, a Traveling Salesperson (TSP) tour is formed through these points to minimize the time required to pickup all these messages. Likewise, once messages are picked up and destination locations are known, a second TSP tour is formed to perform the deliveries. Note that while each vehicle is performing a batch service, new messages are being assigned to it. The vehicle collects this assignment information into a new batch to be served once other outstanding batches have been served.

A more complete description of the policy is given below.

(a) *Message Assignment.* Upon arrival, each message is assigned to the vehicle closest to its source location at the time of the message arrival. All messages assigned to a single vehicle that arrive in the interval $[kT, (k+1)T)$ form a batch, where T, the batch time interval, is a parameter to be determined. Each batch is deposited into a queue for its assigned vehicle upon formation at time $(k+1)T$ for appropriate k.

(b) *Message pickup and delivery.* Batches for each vehicle are served in First Come, First Serve order from the vehicle's batch queue. Pickups are performed along a shortest path through the source locations which is computed at the beginning of the interval. Once pickups are complete, a shortest path through the delivery locations is computed and the deliveries are performed accordingly. To perform each service, the vehicle stops at the source (destination) location for $\frac{n}{\lambda K}$ time to pickup (deliver) the associated message.

The delay analysis of this policy in [1] gives the following result:

Theorem 1(b): The average delay per message for the Source only policy described above scales as:

$$W_{SO} = O\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n}\right) \tag{33}$$

Therefore, the optimal average delay per message over all policies with Source Only information is $\Theta\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n}\right)$.

B. Source-Destination Policy

In the Source-Destination information structure, destination information may be used by vehicles in deciding which messages to pick up. By exploiting this information, vehicles need not traverse the entire geographical region when servicing messages, but may instead only pick up messages that have both source and destination locations in a limited area. For this, a spatially based message assignment policy is used.

That is, in the source destination policy described below, each vehicle is assigned a pickup region and a delivery region. Messages are not assigned to a random vehicle as above, but are instead assigned to the vehicle that has the message's source location in its pickup region and the message's destination location in its delivery region. Again, as above, message assignments are accumulated into batches and batch services are performed along TSP tours through the pickup and delivery regions. Even though the message service policy is similar to that used in the Source Only policy above, Theorem 2(b) shows that the change in assignment policy made possible by using both source and destination information has a significant effect on message delay.

A more complete description of the policy is given below.

(a) *Message Assignment.* Divide the geographical region into an $\sqrt{\frac{A}{\sqrt{n}}} \times \sqrt{\frac{A}{\sqrt{n}}}$ grid of subregions, each of area $\frac{A}{\sqrt{n}}$. To each of the n ordered pairs of subregions, assign exactly one vehicle to service that pair. Each vehicle is assigned to pickup all messages that originate in the first subregion of its assigned ordered pair that have a destination location in second assigned subregion. As before, all messages assigned to a single vehicle that arrive in the interval $[kT, (k+1)T)$ form a batch, where T, the batch time interval, is a parameter to be determined. Each batch is deposited into a queue for its assigned vehicle upon formation at time $(k+1)T$ for appropriate k.

(b) *Message Pickup and Delivery.* As before, batches for each vehicle are served in First Come, First Serve order from the vehicle's batch queue. Batch pickups and deliveries are performed in the same way as in the policy with Source only information with the notable addition of possible interregion travel time between source region and destination region.

The delay analysis of this policy in [1] gives the following result:

Theorem 2(b): The average delay per message for the Source-Destination policy described above scales as:

$$W_{SD} = O\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n^{3/2}}\right) \tag{34}$$

Therefore, the optimal average delay per message over all policies with Source and Destination information is $\Theta\left(\frac{\lambda(n)A}{v^2(1-\rho)^2 n^{3/2}}\right)$.

V. Discussion

In this paper, we have proven lower bounds on the average delay for messages in the DPDP system under two different information constraints. In the case that Source Only information is available, the average delay scales as $\Theta(\frac{\lambda(n)A}{v^2 n})$. In the case that both Source and Destination information is available, the average delay scales as $\Theta(\frac{\lambda(n)A}{v^2 n^{3/2}})$ which is an additional $O(\sqrt{n})$ improvement over the case where only source information is available. From a system design standpoint, these scalings quantify the perfomance improvements achievable by adding additional information gathering capabilities to the vehicles.

The DTRP results in [7], [8] bound the average delay for the each of the pickup and delivery problems as $\Omega(\lambda(n)A/v^2n^2)$. This is an additional $O(\sqrt{n})$ improvement over the full information case we have examined here. We note that as long as vehicles are required to perform physical pickups and deliveries at the source and destination locations, the DTRP lower bound serves as a lower bound on the DPDP problem. It is conjectured that this delay bound can be achieved by removing the restriction that the same vehicle that picks up a message is the one that delivers it. Preliminary results show that this delay bound can indeed be achieved for the pickup and delivery problem in relay networks.

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APPENDIX

Proof of Theorem 4 In this appendix, we prove the modified version of Theorem 2 in [8] as stated in Theorem 4.

The Dynamic Traveling Repairperson (DTRP) problem refers to the following setup: demands arrive to a closed and bounded region $\mathcal{A}$ of area A according to a stationary renewal process. We will assume for our purposes that demands arrive according to a Poisson process with rate λ. This is not as general as the orginal DTRP result in [8], but it is all we require for our DPDP problem. Demands are independently and identically distributed according to the demand distribution $f(x)$. It is assumed that f is K-lipschitz and is bounded from above and below, such that $0 < \underline{f} < f(x) < \bar{f} < \infty, \forall x$. Note that in applying this theorem to the DPDP, we may have $f_i(\zeta)$ such that $f_i(\zeta) = 0$ for some ζ. We may assume for our application that we apply the following analysis only to the regions $\mathcal{A}_i \subset \mathcal{A}$ such that $f_i(\zeta) > 0, \forall \zeta \in A_i$.

There are n vehicles traveling in the region with bounded velocity v to service these demands. A demand is serviced when a vehicle arrives at the demand location and spends a random service time s at that location. The goal is to service the demands with the minimum average delay W between message arrival and service.

Before proving lower bounds on this average delay, [8] provides a few additonal definitions and assumptions. First, with every subset $\mathcal{S} \in \mathbb{R}^2$, associate a queue $\mathcal{S}$, viewed as a black box that has arrivals and departures according to the arrival and service of demands in $\mathcal{S}$. Let $N(\mathcal{S})$ denote the time average number of customers in $\mathcal{S}$ and assume that this time average exists for all $\mathcal{S}$. Let $N = N(\mathcal{A})$ denote the time average number of customers in the whole system.

The conditional waiting time $W(x)$ is defined to be $W(x) \equiv E[W_j | X_j = x]$ where W_j is the waiting time of a demand arriving at X_j . That is, $W(x)$ is the expected waiting time for a demand arriving at x. The normalized waiting time $\Psi(x)$ is then defined to be $\Psi(x) \equiv W(x)/W$. The analysis of [8] is limited to policies such that $\Psi(x)$ is k-lipschitz and bounded from above and below, such that $0 < \underline{\Psi} < \Psi(x) < \bar{\Psi} < \infty, \forall x$. Given these assumptions on the waiting time density, Lemma 1 [8] proves the existence of the queue occupancy density $\phi(x) = f(x)\Psi(x)$ such that

$$\int_{\mathcal{A}} \phi(x)dx = \int_{\mathcal{A}} \frac{f(x)W(x)}{W} dx = 1. \tag{35}$$

Furthermore, if $S \subset \mathcal{A}$, then

$$N(S) = N \int_{S} \phi(x)dx. \tag{36}$$

The proof of Theorem 2 in [8] is outlined as follows. Lemma 2 is unimportant to our analysis since we assume Poisson arrivals. The total service time associated with a demand is defined to be the onsite service time s plus the incremental travel time between the demand and the next demand to be serviced. Denoting the distance to be traveled after the jth demand as d_j, the total service time associated with demand j is then $d_j/v+s_j$. Lemma 3 bounds the average interdemand travel time $\bar{d}$ as

$$\mathbb{E}[Z^*] \equiv \lim_{j\to\infty} \mathbb{E}[Z^*(j)] \leq \lim_{j\to\infty} \mathbb{E}[d_j] \equiv \bar{d}.$$

where $\mathbb{E}[Z^*]$ is the expected minimum distance between any two active demands.

Lemma 4 relates $\mathbb{E}[Z^*]$, the expected minimum distance between active demands, to the system parameters as follows:

$$\lim_{N\to\infty} \sqrt{N}\mathbb{E}[Z^*] \geq \gamma \int_{\mathcal{A}} \phi^{-1/2}(x) f(x) dx \quad (37)$$

where $\gamma \geq \frac{2}{3\sqrt{\pi}}$. Note that as there are more demands are in the queue on average, the interpoint distance decreases, thus decreasing the service time associated with each demand.

Lemma 5 then uses a stability condition to bound the minimum average number in queue, N, required to make the demand services small enough with respect to the arrival rate for stability. Then, Little's Theorem is used to bound $W = N/\lambda$ in terms of $\phi(x)$. A modified proof of this lemma is shown below.

In [8], Lemma 5 is then stated as follows:

Lemma 5 from [8]

$$\lim_{\rho\to 1} W(1-\rho)^2 \geq \gamma^2 \frac{\lambda[\int_{\mathcal{A}} \phi^{-1/2}(x) f(x) dx]^2}{v^2 n^2} \quad (38)$$

As we are interested in the case where $\lambda/n \to \infty$, we instead prove the following result:

Lemma 1: (Lemma 5 from [8] (modified))

For $\frac{\lambda}{n} \to \infty$,

$$W \geq \gamma^2 \frac{\lambda[\int_{\mathcal{A}} \phi^{-1/2}(x) f(x) dx]^2}{v^2(1-\rho)^2 n^2} \quad (39)$$

Proof: Consider the following necessary condition for stability

$$\bar{s} + \frac{\bar{d}}{v} \leq \frac{n}{\lambda}. \quad (40)$$

Using the fact that $\mathbb{E}[Z^*] \leq \bar{d}$, multiplying the second term on the left hand side above by $\frac{\sqrt{N}}{\sqrt{N}}$ and rearranging implies

$$\sqrt{N} \geq \frac{\lambda}{nv(1-\rho)} \sqrt{N}\mathbb{E}[Z^*] \quad (41)$$

Note that from equation (37) above and the definition of the limit, for every $\epsilon > 0$, there exists an N_0 such that for all $N > N_0$,

$$\sqrt{N}\mathbb{E}[Z^*] \geq \gamma \int_{\mathcal{A}} \phi^{-1/2}(x) f(x) dx - \epsilon. \quad (42)$$

Taking $\epsilon = \frac{1}{2}\gamma \int_{\mathcal{A}} \phi^{-1/2}(x) f(x) dx$, we have that

$$\sqrt{N}\mathbb{E}[Z^*] \geq \frac{1}{2}\gamma \int_{\mathcal{A}} \phi^{-1/2}(x) f(x) dx. \quad (43)$$

Substituting this into (41), we have

$$\sqrt{N} \geq \frac{\lambda}{nv(1-\rho)} \frac{\gamma}{2} \int_{\mathcal{A}} \phi^{-1/2}(x) f(x) dx \quad (44)$$

for all $N > N_0$.

The above equation (44) is valid for N sufficiently large. Recall that we are considering the scaling behavior as $\lambda/n \to \infty$. We must then show that $N \to \infty$ as $\lambda/n \to \infty$. Corollary 1 below proves that this is the case.

Therefore we have shown that (44) is valid for λ/n sufficiently large. Squaring both sides of (44) and applying Little's Theorem, $N = \lambda W$, we then have

$$W \geq \gamma^2 \frac{\lambda[\int_{\mathcal{A}} \phi^{-1/2}(x) f(x) dx]^2}{v^2(1-\rho)^2 n^2} \quad (45)$$

and the modified lemma is proven.

■

To complete the proof of Theorem 4, we use the proof of Theorem 2 in [8] as originally written. Since $\phi(x)$ is policy dependent, this proof solves for $\min[\int_{\mathcal{A}} \phi^{-1/2}(x) f(x) dx]^2$ as a function of $f(x)$. Theorem 4 here differs from Theorem 2 in [8] only in the restatement of the limiting terms as in the modified lemma.

Finally, to complete our modifed proof of Lemma 5, we must show that $N \to \infty$ as $\lambda/n \to \infty$ in the DTRP system. We first prove a preliminary lemma on the scaling of the system workload in a DTRP queue where workload is defined as in the standard definition of workload in the context of networks:

Definition 1 (Workload): The workload in the system at time t, $V(t)$, is the amount of time it takes the n vehicles to serve all of the messages currently in the system at time t.

To show that the average work in system goes to ∞ as $\lambda/n \to$

∞, we have the following lemma.

Lemma 2: The average workload in the system V is lower bounded by:

$$V \geq c \frac{\lambda \mathbb{E}[\sqrt{f}]^2}{n} \quad (46)$$

where $c = \frac{\beta_{TSP}^2(\sqrt{2}-1)}{64} \approx 0.0064\beta_{TSP}^2$.

Proof: Assume the vehicle started serving at time $-\infty$. Now consider any time, say 0. Let $V(0)$ denote the amount of workload in the system at time 0. Since time 0 is arbitrary, $V(0)$ is distributed like the stationary distribution of workload. Let $A(s)$ denote the minimal amount of time it takes to serve messages arriving in interval $[-t, 0]$. Then, it is easy to see that

$$V(0) \quad \geq \quad (A(t) - nt)^+ . \quad (47)$$

That is, the work in system is greater than difference between the amount of arrived work in an interval of length t and the maximum possible work completed by the n vehicles in the interval. The equation (47) is true for all t. Further, the time 0 is a randomly chosen time and hence represents the stationary time. Hence, we obtain the time average of workload in the system, $\mathbb{E}[V]$, is lower bounded as

$$\mathbb{E}[V] \quad \geq \quad \mathbb{E}[A(t)] - nt, \ \forall t \geq 0. \quad (48)$$

Thus, to compute lower bound on average workload V, we need to compute $\mathbb{E}[A(t)]$. That is, we need to compute the average minimal time required to serve messages arriving to the system in an interval of length t. Now, recall that demands arrive according to the distribution $f(x)$ and that we assume a Poisson arrival process of rate λ. Let $N(t)$ be random number of arrivals happening in time interval of length t. Then, $A(t)$ can be lower bounded by the length of shortest path connecting all source and destination locations of these $N(t)$ messages. Now, the length of a shortest path through a set of locations is no longer than twice the length of the shortest cycle through these points, the TSP tour. Similarly, note that the TSP tour is no more than twice the length of the shortest path through these points. Hence, to obtain lower bound $A(t)$, it is sufficient to consider the length of TSP tour through the source and destination location of $N(t)$ points.

The following is a well-known Beardwood, Halton, and Hammersley [10] bound on the length of a TSP tour.

Theorem 5: (BHH Theorem)

Let L_N denote the length of the TSP tour through N points. For $N \to \infty$, if point are distributed according to probability density $f(\cdot)$, then

$$\mathbb{E}[L_N] \approx \beta_{TSP}\sqrt{N}\mathbb{E}[\sqrt{f}], \quad (49)$$

where β_{TSP} is a finite positive constant.

Note that $\mathbb{E}[N(t)] = \lambda t$. Therefore, as $t \to \infty$, $\mathbb{E}[N(t)] \to \infty$ and the BHH result becomes tight. Also due to the Poisson property of the arrival process, $N(t) \geq \lambda t/2$ with probability at least $1/2$ for large enough λ. Therefore, $P(\sqrt{N(t)} \geq \sqrt{\lambda t/2}) \geq 1/2$ and

$$\mathbb{E}[\sqrt{N(t)}] \geq \frac{1}{2}\sqrt{\frac{\lambda t}{2}}. \quad (50)$$

Assuming that t is sufficiently large so that Theorem 5 is tight and substituting in (50), we may lower bound A as

$$\mathbb{E}[A(t)] \quad \geq \quad \hat{\beta}\sqrt{\lambda t}\mathbb{E}[\sqrt{f}], \quad (51)$$

where $\hat{\beta} = \frac{\beta_{TSP}}{4\sqrt{2}}$. From (48) and (51), we obtain

$$\mathbb{E}[V] \quad \geq \quad \hat{\beta}\sqrt{\lambda t}\mathbb{E}[\sqrt{f}] - nt. \quad (52)$$

Consider $t^* = \frac{\lambda \hat{\beta}^2 \mathbb{E}[\sqrt{f_i}]^2}{2n}$. Note that as $\lambda/n \to \infty$ this t is sufficiently large as required for the tightness of Theorem 5. Then, from (52) we obtain

$$\mathbb{E}[V] \quad \geq \quad \frac{\sqrt{2}-1}{2}\frac{\hat{\beta}^2\lambda\mathbb{E}[\sqrt{f}]^2}{n}. \quad (53)$$

and Lemma 2 is proven. ■

Corollary 1: As $\lambda/n \to \infty$, the average number in queue $N \to \infty$.

Proof: First use Lemma 2 to show that the average workload $V \to \infty$. By assumption, $\mathbb{E}[\sqrt{f}] > 0$. Since the arrival distribution $f(x)$ does not depend on λ or n and $\mathbb{E}[\sqrt{f}]^2$ is bounded away from 0, as $\lambda/n \to \infty$, the workload $V \geq c\frac{\lambda\mathbb{E}[\sqrt{f}]^2}{n} \to \infty$ as well.

Now note that the work associated with each message is upper bounded by the diameter of the region plus the onsite service time, $\sqrt{2}\sqrt{A} + \frac{n}{\lambda K}$. Therefore, if the average work in the system is going to ∞ and the work associated with each message is finite, the average number of messages in the system, N, must be going to ∞ as well.

■

This completes the proof of Theorem 4.

■