Product Multicommodity Flow in Wireless Networks

Ritesh Madan, Devavrat Shah, and Olivier Lévêque, *Member, IEEE*

***Abstract*—We provide a tight approximate characterization of the n-dimensional product multicommodity flow (PMF) region for a wireless network of n nodes. Separate characterizations in terms of the spectral properties of appropriate network graphs are obtained in both an information-theoretic sense and for a combinatorial interference model (e.g., Protocol model). These provide an inner approximation to the n^2-dimensional capacity region. Our results hold for general node distributions, traffic models, and channel fading models.**

We first establish that the random source–destination model assumed in many previous results on capacity scaling laws, is essentially a one-dimensional approximation to the capacity region and a special case of PMF. We then build on the results for a wireline network (graph) that relate PMF to its spectral (or cut) properties. Specifically, for a combinatorial interference model given by a network graph and a conflict graph, we relate the PMF to the spectral properties of the underlying graphs resulting in simple computational upper and lower bounds. These results show that the $1/\sqrt{n}$ scaling law obtained by Gupta and Kumar for a geometric random network can be explained in terms of the scaling law of the *conductance* of a geometric random graph. For the more interesting random fading model with additive white Gaussian noise (AWGN), we show that the scaling laws for PMF can again be tightly characterized by the spectral properties of appropriately defined graphs—such a characterization for general wireless networks has not been available before. As an implication, we obtain computationally efficient upper and lower bounds on the PMF for any wireless network with a guaranteed approximation factor.

***Index Terms*—Capacity region, product multicommodity flow (PMF), scaling law, wireless network.**

I. Introduction

A. Prior Work

An important open question in network information theory is that of characterizing the capacity region of a wireless network of n nodes, i.e., the set of all achievable rates between the n^2 pairs of nodes in terms of the joint statistics of the channels between these nodes. This has proved to be a very challenging question; even the capacity of a relay network composed of three nodes is not known in complete generality.

Instead of trying to characterize the capacity region for a general wireless network, the seminal paper by Gupta and Kumar [1] concentrated on obtaining the maximum achievable rate for a particular communication model, geometric random distribution of nodes, and randomly chosen source–destination pairs. They showed that the maximum rate for the protocol interference model scales as $\Theta(1/\sqrt{n})$ for n nodes randomly placed on a sphere of unit area. This characterization has been followed by many interesting results for both combinatorial interference models and the random fading information-theoretic model for large random networks; these include [2]–[7] for communication-theoretic models, and [8] for information-theoretic results. These results are crucially based on the assumption that a large number of nodes are randomly distributed in a certain region, and on the inherent symmetry in the random source–destination pair traffic model. Such scaling laws are interesting because they provide a simple characterization of the maximum achievable rate in terms of the number of nodes in the network.

Since the relative locations of wireless nodes play an important role in the characterization of the capacity region, the notion of transport capacity was defined in [1]. A scaling law for the transport capacity for the protocol interference model was obtained in [1]. Random fading was considered in [7], and information-theoretic upper bounds were obtained in [9], [10]. The transport capacity can be used to obtain an upper bound on the achievable rate-region for certain rate-tuples, but is not of much use in determining the feasibility of a certain rate-tuple. More recently, information-theoretic outer bounds to the capacity region of a wireless network with a finite number of nodes were obtained in [11] for any wireless network using the cut-set bound [12, Ch. 14]. We note that any achievable scheme can be used to obtain a set of lower bounds. For example, the results in [13] provide one set of such bounds. While the above is only a discussion of a representative set of results in this area (see [14] for a more detailed summary), we note that there is no result which provides upper and lower bounds with a guaranteed approximation factor for a general wireless network with a generic random fading model. In this paper, we take the first steps towards providing such a tight characterization under very general assumptions. In doing so, we make connections between spectral graph-theoretic results and network information theory. This results in efficient methods to compute tight upper and lower bounds.

B. Contribution and Organization

In Section II, we consider the product multicommodity flow (PMF) as an n-dimensional approximation of the n^2-dimensional capacity region. We show that the random souce–desti-

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R. Madan is with QUALCOMM-Flarion Technologies, Bridgewater, NJ 08807 USA (e-mail: rkmaden@stanfordalumni.org).

D. Shah is with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: devavrat@mit.edu).

O. Lévêque is with the Ecole Politechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland (e-mail: olivier.leveque@epfl.ch).

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nation pair traffic model is a special case of PMF and it is essentially a one-dimensional approximation of the capacity region.

In Section III, we study the PMF for an arbitrary topology and a general combinatorial interference model, of which the protocol model is a special case. We show that the normalized cut capacity (equivalently, conductance) of a capacitated network graph induced by the node placement and the interference model characterizes the PMF within a $\log n$ factor. Using elementary arguments which are independent of the node distribution and the path loss model, we also obtain a scaling law for delay. We provide simpler rederivation of the (weaker by $\log^{2.5} n$ factor) lower bound on the maximum flow obtained by Gupta and Kumar for a randomly chosen permutation flow on a geometric random graph with a protocol interference model. For this, we evaluate the scaling law for the conductance of a geometric random graph which is new and interesting in its own right. Our derivation illustrates the connections between the combinatorial properties of geometric random graphs and the maximum PMF.

In Section IV, we address the question of characterizing the PMF for a wireless network with Gaussian channels and random fading. This is substantially more challenging than for the combinatorial interference model because there is no obvious underlying network graph that specifies the links which should be used for data transmission. We construct a capacitated graph whose cut capacity characterizes (in terms of tight upper and lower bounds) the PMF in the wireless network. This construction allows one to use classical network flow arguments to characterize and compute the PMF. We illustrate the generality of our results by obtaining scaling laws for a geometric random network.

II. Traffic Flows

In this section, we describe the class of PMFs and its relevance. Consider a wireless network of n nodes and denote the node set as $V = \{1, \dots, n\}$. A traffic matrix $\lambda = [\lambda_{ij}] \in \mathbb{R}_+^{n\times n}$ is said to be *feasible* if for each pair of nodes (i, j), $1 \le i, j \le n$, data can be transmitted from node i to node j at rate λ_{ij}. Note that whether a traffic matrix λ is feasible or not depends on the model for the underlying wireless network, and we shall describe the precise models for wireless networks in the later sections.

We denote the capacity region by Λ, i.e., Λ is the set of all feasible traffic matrices. Ideally, we would like to characterize Λ. However, this is a hard problem in most cases. Instead, we characterize an *approximation* of Λ under general assumptions on the wireless network. For this, we consider PMF defined as follows.

Definition 1 (Product Multicommodity Flow (PMF)): Let node i be assigned a weight $\pi(i) \ge 0$, for $1 \le i \le n$. Then the PMF corresponding to the weights $\pi \in \mathbb{R}_+^n$ and a flow rate $f \in \mathbb{R}_+$ is given by the function [15]$M : \mathbb{R}^{n+1} \mapsto \mathbb{R}^{n\times n}$:

$$M(f,\pi) = f \begin{bmatrix} 0 & \pi(1)\pi(2) & \dots & \pi(1)\pi(n) \\ \pi(2)\pi(1) & 0 & \dots & \pi(2)\pi(n) \\ \vdots & \vdots & \vdots & \vdots \\ \pi(n)\pi(1) & \pi(n)\pi(2) & \dots & 0 \end{bmatrix}.$$

The PMF is an n-dimensional approximation to the n^2-dimensional capacity region Λ with *product* constraints. An important special case arises when all the weights are 1, i.e., $\pi(i) = 1$ for $i = 1, \dots, n$. We call such a flow uniform multicommodity flow (UMF).

Definition 2 (Uniform Multicommodity Flow (UMF)): UMF with flow rate $f \in \mathbb{R}_+$, denoted by $U(f)$, is an $n \times n$ matrix with diagonal entries equal to 0, and all nondiagonal entries equal to f.

We denote by f_π^* the supremum over the flow rates for which the PMF corresponding to the weights π is feasible, i.e.,

$$f_\pi^* = \sup\{f \in R_+ : M(f,\pi) \text{ is feasible}\}.$$

We abuse notation and denote the corresponding quantity for UMF as simply f^*.

A. Inner Approximation to Λ

We first show that the maximum UMF f^* is a one-parameter approximation to the capacity region Λ. Consider the following parameter defined in terms of the capacity region.

Definition 3 (ρ^*): For any $\lambda \in \mathbb{R}_+^{n\times n}$, let

$$\rho(\lambda) \triangleq \max_i \left\{ \sum_{k=1}^n \lambda_{ik}, \sum_{k=1}^n \lambda_{ki} \right\}.$$

Also, let

$$L(x) = \{\lambda \in \mathbb{R}_+^{n\times n} : \rho(\lambda) \le x\}.$$

Then, we define ρ^* as follows:

$$\rho^* = \sup\{x \in \mathbb{R}_+ : L(x) \subseteq \Lambda\}.$$

Thus, the quantity ρ^* is a parametrization of a polyhedral inner approximation to the capacity region Λ. It is tight in the sense for any $x > \rho^*$, there is an infeasible traffic matrix in the set $L(x)$.

Roughly speaking, the following result shows f^* and ρ^* are equally good approximations to the capacity region Λ.

Lemma 1: If $U(f)$ is feasible, then any $\lambda \in \mathbb{R}_+^{n\times n}$ such that $\rho(\lambda) \le nf/2$ is feasible.

Proof: Consider any λ such that $\rho(\lambda) \le nf/2$. Suppose that $U(f)$ is feasible. Then there exists a transmission scheme which supports $U(f)$. We now consider the two-stage routing scheme of Valiant and Brenber [16] which routes $U(\rho(\lambda)/n)$ in each stage. In the first stage, each node i sends data to all the remaining nodes uniformly, ignoring its actual destination. Thus, node i sends data to any node j at rate $\sum_k \lambda_{ik}/n \le \rho(\lambda)/n$. In the second stage, a node, say j, on receiving data (from the first stage) from any source i sends it to the appropriate destination. It is easy to see that due to the uniform spreading of data in the first stage, each node j routes data at rate $\sum_k \lambda_{ki}/n \le \rho(\lambda)/n$ to node i in the second stage. Thus, the traffic matrices routed in both the stages are dominated by $U(\rho(\lambda)/n)$. That is, the sum traffic matrix is dominated by $U(2\rho(\lambda)/n)$. Hence, if $U(f)$ is feasible then $\rho(\lambda) \le nf/2$ is feasible using time sharing between the schemes corresponding to the two stages above. This completes the proof of Lemma 1. □

Theorem 1: f^* and ρ^* are related as

$$\frac{nf^*}{2} \leq \rho^* \leq nf^*.$$

Proof: Note that in general, the capacity region Λ may not be a closed set.[1] We first show that $\frac{nf^*}{2} \leq \rho^*$. By definition of sup it follows that for any $\epsilon > 0$, $U(f^* - \epsilon/(2n))$ is feasible. Hence, from Lemma 1, any $\lambda \in \mathbb{R}_+^{n\times n}$ such that $\rho(\lambda) \leq nf^*/2 - \epsilon$ is feasible. Hence, again using the definition of sup, $\rho^* \geq nf^*/2$.

Now for the other bound, assume that $\rho^* > nf^*$ and $\epsilon = (\rho^* - nf^*)/2$. Then, by definition of sup and $\rho, U(nf^* + \epsilon/2)$ is feasible, which is a contradiction. Hence, it follows that $\rho^* \leq nf^*$. □

Thus, bounds on f^* give bounds on ρ^* which differ by at most a factor of 2. Note that this factor is independent of any network model. Subsequently, a scaling law for f^* as a function of n is the same as a scaling law for ρ^*, i.e., $f^* = \Theta(\rho^*)$ as a function of n.

The set of all feasible PMFs clearly provides an n-dimensional inner approximation to the capacity region, which is, in general, n^2-dimensional. Thus, the characterization of the set of feasible PMFs provides a much better approximation to the capacity region than that the one-dimensional approximation given by set of feasible UMF. We next establish the equivalence of UMF and a traffic model with a randomly chosen permutation flow.

1) UMF and Random Permutation Flow: In some previous work (e.g., [1]), the capacity scaling laws were derived for the case where n distinct source–destination pairs are chosen at random such that each node is a source (destination) for exactly one destination (source) and such a pairing is done uniformly at random over all possible such pairings. Thus, the traffic matrix corresponds to a randomly chosen permutation flow which is defined as follows.

Definition 4 (Permutation Flow): Let S_n denote the set of permutation matrices in $\mathbb{R}_+^{n\times n}$. Then the permutation flow corresponding to a permutation $\Sigma \in S_n$ and flow rate $f \in \mathbb{R}_+$ is given by $S(f, \Sigma) = f\Sigma$.

Many previous works study the scaling of $\bar{f}$, where $\bar{f}$ is the supremum over the set of $f \in \mathbb{R}_+$ such that when a permutation Σ is randomly chosen from S_n, the permutation flow $S(f, \Sigma)$ is feasible with probability at least $1 - 1/n^2$. We now show that when a permutation flow with flow rate nf and a randomly chosen permutation is feasible with a high enough probability, then the uniform mulicommodity flow $U(f)$ can be "almost" supported when n is large enough.

Lemma 2: For $\Sigma \in S_n$ chosen uniformly at random, if $(nf)\Sigma$ is feasible with probability at least $1 - n^{-1-\alpha}$, $\alpha > 0$, then there exists a sequence of feasible rate matrices Γ_n such that[2]

$$\|U_n(f) - \Gamma_n\| = O(fn^{-\alpha}) \to 0 \text{ as } n \to \infty$$

where $\|\cdot\|$ denotes the standard 2-norm for matrices,[3] and $U_n(f)$ is the uniform multicommodity flow for n nodes.

Proof: See Appendix A. □

From Lemma 1, if $U(f)$ is feasible, then $S(nf/2, \Sigma)$ is feasible for all $\Sigma \in S_n$. Thus, using an argument identical to that in the proof of Theorem 1, a scaling law for $\bar{f}$ is equivalent to a scaling law for f^*, i.e.,

$$f^* = \Theta(n\bar{f}).$$

B. Wireline Networks: PMF Over a Graph

We briefly review the key known results for PMF on graphs with fixed edge capacities. These results will be useful in our analysis for PMF for wireless networks.

Consider a directed graph $G = (V, E)$, where an edge $(i, j) \in E$ has a capacity $C(i, j)$. Also, for $(i, j) \notin E$, we take $C(i, j) = 0$. Then for a given π, f_π^* for graph $G = (V, E)$ is given by the solution of the following linear program (LP):

$$\begin{aligned} \max.\ & f, \\ \text{s.t.}\ & \sum_{k:(i,k)\in E} x_{ij}(i,k) - \sum_{k:(k,i)\in E} x_{ij}(k,i) = f\pi(i)\pi(j), \\ & \qquad i \neq j, 1 \leq i, j \leq n \\ & \sum_{m:(k,m)\in E} x_{ij}(k,m) - \sum_{m:(m,k)\in E} x_{ij}(m,k) = 0, \\ & \qquad \forall\, k \neq i, j,\ 1 \leq i, j \leq n \\ & \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}(k,m) \leq C(k,m), \quad \forall\,(k,m) \in E \end{aligned}$$

where the variables are f and $\{x_{ij}(k,m) : (k,m) \in E, i, j, k, m = 1, \ldots, n\}$. The first two sets of constraints are flow conservation constraints and the third set of constraints model the finite capacity at each edge. The total number of variables is less than $2n^4$ and the total number of constraints is less than $(n^3 + 2n^2)$. Hence, the above LP can be solved in $O(n^{12})$ time even if the structure of the problem is not exploited [17].

The well-known max-flow min-cut characterization for a single commodity flow naturally gives rise to the following question. Though the maximum PMF f_π^* for a given weight vector can be computed in polynomial time, is there a corresponding result that relates f_π^* and the properties of the graph? In their seminal paper, Leighton and Rao [18] obtained a characterization of f_π^* in terms of the weighted min-cut of graph. We summarize their main result below. Let $p_\pi = |\{i \in V : \pi(i) > 0\}|$ denote the number of nodes for which the corresponding element of π is nonzero. Then, without loss of generality we assume that $\sum_{i=1}^n \pi(i) = p_\pi$.

Definition 5: For graph G and weight vector π, define the min-cut by

$$\Psi_\pi(G) = \min_{U \subseteq V} \frac{\sum_{(i,j): i\in U, j\in U^C} C(i,j)}{\pi(U)\pi(U^c)}$$

where $\pi(S) = \sum_{i\in S} \pi(i)$ for all $S \subseteq V$.

[1]We present a formal argument to deal with this only once; similar arguments are implicit in many results that follow.

[2]In this paper, the O and Ω notation is always with respect to the number of nodes n in the network.

[3]Given a matrix $M \in \mathbb{R}^{n\times n}$, the 2-norm of M is $\|M\| = \sup\{\|Mx\| : x \in \mathbb{R}^n, \|x\| = 1\}$, where $\|x\|$ is the ℓ_2 norm of vector $x \in \mathbb{R}^n$.

Theorem 2 ([15, Theorem 17]): In any directed graph G, the maximum PMF for weight π is related to $\Psi_\pi(G)$ as follows:

$$\Omega\left(\frac{\Psi_\pi(G)}{\log p_\pi}\right) \le f_\pi^* \le \Psi_\pi(G)$$

where the constants for the lower bound do not depend on the graph.

Note that the upper bound follows easily because for a given PMF f_π, the total flow from U to U^C is $\pi(U)\pi(U^C)f_\pi$, which has to be less than the sum capacity of the links from U to U^C. The above characterization was crucial to the design of subsequent approximation algorithms for many NP-hard problems; a summary of these algorithms can be found in [15]. An important case of the above result is when $\pi(i) = 1$ for all $i = 1, \ldots, n$, i.e., the special case of uniform mulitcommodity flow. In this case, we have

$$\Psi_{\mathbf{1}}(G) = \min_{U \subseteq V} \frac{\sum_{(i,j):i\in U, j\in U^C} C(i,j)}{|U||U^C|}$$

and

$$\Omega\left(\frac{\Psi_{\mathbf{1}}(G)}{\log n}\right) \le f^* \le \Psi_{\mathbf{1}}(G).$$

As we will see, using such characterizations we are able to evaluate simple closed-form scaling laws for random networks and provide computational lower and upper bounds for general networks. The lower bounds are constructive and the approximation factor can be easily characterized.

III. Combinatorial Interference Model

A combinatorial interference model defines constraints such that simultaneous data transmissions over only certain sets of links (or edges) can be successful. This is a simplified abstraction of a wireless network because in reality whether or not multiple simultaneous data transmissions are successful depends on the rate of data transmission and the interference power at the various receivers. We first describe the combinatorial interference model formally and illustrate it with example scenarios where this abstraction is a reasonable one.

A. Model

A combinatorial interference model for a given set of wireless nodes $V = \{1, \ldots, n\}$ defines the following two objects.

(a) A directed graph $G = (V, E)$ where E is the set of directed links (edges) over which data can be transmitted.

(b) For each directed edge $e \in E$, $\mathcal{I}(e) = \{\hat{e} \in E\}$ is the set of edges (directed links) that interfere with a transmission on link e. Data can be successfully transmitted on link e at rate $W(e)$ if and only if no transmission on any link in $\mathcal{I}(e)$ takes place simultaneously. In general, the rate $W(e)$ for a given power constraint can be different for different edges. The proof methods and results in this paper extend easily to this general case. However, for the ease of exposition we will assume $W(e) = 1$[4] for all $e \in E$.

[4]As long as $W(e)$ is bounded below and above by a constant, scaling laws do not change even though the bounds for a given number of nodes n will change.

We assume that for every edge $(i,j) \in E$, edge $(j,i) \in E$, i.e., the graph G is essentially an undirected graph without the interference constraints given by the sets $\mathcal{I}(e)$'s. This is a reasonable assumption in many time-division and frequency-division systems, where the channels are reciprocal [19]. The interference sets $\mathcal{I}((i,j))$ and $\mathcal{I}((j,i))$ may not be identical because the transmissions which interfere with a signal received at node i may not be the same as transmissions which interfere with a signal received at node j.

The above definitions can be used to induce a dual conflict graph as follows.

Definition 6: The dual conflict graph is a undirected graph $G^D = (E, E^D)$ with vertex set E and edge set E^D, where an edge $e^D \in E^D$ exists between e_1 and e_2 if e_1 and e_2 cannot transmit simultaneously due to interference constraints. Thus, each link $e \in E$ is connected to all links in $\mathcal{I}(e)$.

For the rest of the section, to simplify notation, we will suppress the explicit dependence of all quantities on the combinatorial interference model parameterized by the graphs G and G^D. Let us denote the node degree and the chromatic number[5] of the dual conflict graph G^D by $\Delta = \max_{e\in E} |\mathcal{I}(e)|$ and κ, respectively. Note that $\kappa \le (1 + \Delta)$. Let $\{E_k\}$, $E_k \subseteq E$, be the set of all possible link sets that can be active simultaneously, i.e., simultaneous transmissions on all the links in E_k at rate $W(e) = 1$ are feasible for the given interference model. Each E_k corresponds to a vector $C_k \in \mathbb{R}^{|E|}$, where $C_k(e) = \mathbf{1}_{\{e\in E_k\}}$. Let $\mathcal{C}$ be the convex hull of all such vectors $\{C_k\}$. Thus, $\mathcal{C}$ is the set of all vectors C such that link capacities $C(e)$ (for link e) can be obtained by time sharing between the C_k's for the given interference model. We then define the capacity region to be the set of traffic matrices which can be routed over the graph $G = (V, E)$ such that each edge e has capacity $C(e)$, for some $C \in \mathcal{C}$. The formal definition is as follows.

Definition 7 (Capacity Region (Λ)): The capacity region is the set of traffic matrices $\lambda \in \mathbb{R}^{n\times n}$ such that the following set of conditions are feasible for some $C \in \mathcal{C}$:

$$\sum_{k:(i,k)\in E} \Big(x_{ij}(i,k) - x_{ij}(k,i)\Big) = \lambda_{ij}, \quad i \ne j,\ 1 \le i,j \le n$$

$$\sum_{m:(k,m)\in E} \Big(x_{ij}(k,m) - x_{ij}(m,k)\Big) = 0, \qquad \forall\, k \ne i,j,\ 1 \le i,j \le n$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}(k,m) \le C(k,m),\ \forall (k,m) \in E \tag{1}$$

where $C(k,m) = C(e)$ for $e = (k,m) \in E$; the variables are $\{x_{ij}(k,m) : (k,m) \in E, 1 \le i,j,k,m \le n\}$.

Thus, the capacity region consists of all traffic matrices which can be supported using a transmission scheme which is a combination of routing and link scheduling (time-sharing between the sets $\{E_k\}$). We now illustrate this capacity region by a couple of special cases corresponding to widely used models for wireless networks.

[5]The chromatic number of a graph is the minimum number of colors needed to color the nodes of the graph such that no two nodes which are connected by an edge share the same color.

1) Protocol Model: The protocol model parameterized by the maximum radius of transmission r, and the amount of acceptable interference η, is defined in [1] as follows.

(a) A node i can transmit to any node j if the distance between i and j, r_{ij}, is less than the transmission radius r.

(b) For transmission from node i to j to be successful, no other node k within distance $(1+\eta)r_{ij}$ ($\eta > 0$, a constant) of node j should transmit simultaneously.

The corresponding definitions of E and E_D follow. A directed link from node i to node j is in E if $r_{ij} \leq r$. For a link $e \in E$, let e^+ denote the transmitter and let e^- denote the receiver. Then

$$\mathcal{I}(e) = \{\hat{e} \in E : r_{\hat{e}^+ e^-} \leq (1+\eta) r_{e^+ e^-}\}.$$

Thus, the protocol model is a special case of the combinatorial interference model.

2) Signal-to-Interference-and-Noise Ratio (SINR) Threshold Model: Assume that all transmissions occur at power P, and the channel gain from the transmitter of node j to the receiver of node i is given by h_{ij}, i.e., if node j transmits at power P, the received signal power at node i will be $h_{ij}P$. A signal-to-interference-and-noise ratio (SINR) threshold model is parametrized by a threshold α such that a transmission from node i to node j is successful if and only if the SINR is above α, i.e.,

$$\frac{Ph_{ij}}{\sum_{k \neq i} Ph_{kj} + N_0 B} \geq \alpha.$$

For example, if we assume that each link transmits Gaussian signals and that the Shannon capacity on each link is achievable, then the threshold is given by $(2^W - 1)$ (assuming $W(e) = W$ for all $e \in E$ as before).

We can define a corresponding combinatorial interference model such that the feasible simultaneous transmissions defined by the combinatorial interference model are a subset of that described by the SINR threshold model. Consider the set of directed links E_γ such that a link (i,j) from node i to node j is in E_γ if and only if $h_{ji} \geq \gamma$. Also, define $\mathcal{I}(e) = \{\hat{e} \in E_\gamma : h_{e^- \hat{e}^+} \geq \beta\}$. Then link e can transmit at rate W if no other links in $\mathcal{I}(e)$ transmit simultaneously, if and only if γ and β are such that

$$\frac{Ph_{e^- e^+}}{\sum_{\hat{e} \in E_\gamma,\ \hat{e} \notin \mathcal{I}(e)} Ph_{e^- \hat{e}^+} + N_0 B} \geq \alpha, \quad \forall\, e \in E_\gamma. \tag{2}$$

It is easy to see that the above condition is satisfied if the following condition holds:

$$\beta \leq \frac{1}{nP}\left(\frac{P\gamma}{2^W - 1} - N_0 B\right).$$

B. Results

We now derive results for the combinatorial interference model which relate the maximum PMF f_π^* to spectral properties of the underlying graphs induced by the interference model. Most of the results in this subsection use ideas from known results. While important in their own right, these results and their proofs motivate the results for an information-theoretic setting for wireless networks with Gaussian channels. Also, they provide alternate derivations for known capacity scaling laws in random networks. In doing so, we derive a scaling law for the conductance of a geometric random graph. This should be of interest in its own right. Towards the end of this subsection, we obtain simple bounds on the delay in terms of the hop-count. Hop-count and delay are equivalent measures for a class of network models as discussed later in this subsection.

1) Bounds on PMF: For any $C \in \mathcal{C}$, we denote the maximum PMF on graph G, where each edge e has capacity $C(e)$, by $f_\pi(C)$, and the corresponding min-cut by

$$\Psi_\pi(C) = \min_{S \subset V} \frac{\sum_{(i,j): i \in S, j \in S^c} C(i,j)}{\pi(S)\pi(S^c)}.$$

We denote the corresponding quantities for the special case of UMF by $f(C)$ and $\Psi(C)$, respectively. Then we have the following lemmas.

Lemma 3: $\psi_\pi : \mathcal{C} \mapsto \mathbb{R}$ is a continuous function for $\{\pi : \pi \geq 0, \max_{S \subset V} \pi(S)\pi(S^c) > 0\}$.

Proof: See Appendix A. □

Lemma 4: $f_\pi : \mathcal{C} \mapsto \mathbb{R}$ is a continuous function for $\{\pi : \pi \geq 0, \max_{S \subset V} \pi(S)\pi(S^c) > 0\}$.

Proof: See Appendix A. □

We now define a quantity for the combinatorial interference model corresponding to the min-cut of a graph.

Definition 8: The min-cut for the combinatorial interference model is defined as

$$\Psi_\pi^* = \max_{C \in \mathcal{C}} \min_{S \subset V} \frac{\sum_{(i,j): i \in S, j \in S^c} C(i,j)}{\pi(S)\pi(S^c)}.$$

Note that Ψ_π^* is well defined since $\Psi_\pi(C)$ is a continuous function of C, and $\mathcal{C}$ is closed and bounded because it is the convex hull of a finite number of points. The above definition can be interpreted as the min-cut of the graph G, where each edge has capacity $C(e)$, and the vector C is chosen from the set $\mathcal{C}$ such that it maximizes the min-cut of this graph G. The following result is an extension of Theorem 2 to combinatorial interference models.

Theorem 3: f_π^* is bounded as

$$\Omega\left(\frac{\Psi_\pi^*}{\log p_\pi}\right) \leq f_\pi^* \leq \Psi_\pi^*. \tag{3}$$

Proof: Since $\mathcal{C}$ is closed and bounded, it follows from Lemma 4 that there exists $C^* \in \mathcal{C}$ such that $f_\pi^* = f_\pi(C^*)$. Then, using Theorem 2, it follows that

$$f_\pi^* \leq \Psi_\pi(C^*) \leq \Psi_\pi^*.$$

Now, from Lemma 3, there is $\hat{C} \in \mathcal{C}$ such that $\Psi_\pi^* = \Psi_\pi(\hat{C})$. Using Theorem 2, it follows that

$$f_\pi^* \geq f_\pi(\hat{C}) = \Omega\left(\frac{\Psi_\pi^*}{\log p_\pi}\right).$$

This completes the proof of Theorem 3. □

Note that unlike the case for wireline networks (or equivalently graphs), f^* is a quantity hard to compute. Also, note that

Ψ is a function of both G and the dual graph G^D. We next relate the maximum UMF f^* to spectral properties of graphs G and G^D.

Definition 9: The conductance of graph G is defined as follows:

$$\Phi = \min_{U \subseteq V, |U| \leq n/2} \frac{\sum_{i \in U, j \in U^C} \mathbf{1}_{[(i,j) \in E]}}{|U|},$$

where $\mathbf{1}_{[.]}$ is the indicator function.

Corollary 1: Recall that κ is the chromatic number of the dual graph G^D. Then, f^* is related to Φ as follows:

$$\Omega\left(\frac{\Phi}{\kappa n \log n}\right) \leq f^* \leq \frac{\Phi}{n}.$$

Proof: Consider vertex coloring for the dual graph $G^D = (E, E^D)$. The chromatic number of G^D is defined to be κ and hence we need κ colors for vertex coloring of G^D. Thus, we have partitioned the set E into subsets, say, $E_1, \ldots, E_\kappa$, such that the links in each subset can transmit simultaneously at rate 1. Now let $C_k(e) = \mathbf{1}_{\{e \in E_k\}}$. Then, C corresponding to uniform time sharing between the κ edge sets $E_1, \ldots, E_k$ is given by

$$C = \frac{1}{\kappa}(C_1 + \cdots + C_\kappa)$$

which is a convex combination of $C_1, \ldots, C_\kappa \in \mathcal{C}$. Hence, $C(i,j) = 1/\kappa$ for all $i \neq j$, and $C \in \mathcal{C}$. Then, using Theorem 2 and the definition of conductance above

$$f^* \geq f^*(C) = \Omega\left(\frac{\Psi(C)}{\log n}\right) = \Omega\left(\frac{\Phi(G)}{\kappa n \log n}\right). \qquad \square$$

For the upper bound, note that for any $C \in \mathcal{C}$, $C \preceq \mathbf{1}$, i.e., C is lexicographically less than $\mathbf{1}$, and $f(C_1) \geq f(C_2)$ if $C_1 \geq C_2$. Hence, $f^* = \max_{C \in \mathcal{C}} f(C) \leq f(\mathbf{1})$. Then, the upper bound follows again by a straightforward use of Theorem 2 and the definition of conductance.

2) Average Delay: We now provide bounds on the average delay for a class of traffic matrices. We measure delay in number of hops, $H(n)$. The number of hops and the actual packet delay are closely related notions. We explain this briefly as follows: If the packet size is small enough, then by appropriate time division of the capacity at nodes, the packet delay becomes essentially equal to the number of hops taken by the packet. This is similar to the model in, for example, [1], [2], and [4]. On the other hand, if packet sizes are constant, then one needs a clever scheduling scheme at the nodes in order to establish that the delay is of the same order as the number of hops for a class of networks in which each server or node gets to transmit once in a constant number of time slots. A detailed analysis for this case for grid networks can be found in [20].

We now characterize the number of hops. In order to do so, we restrict ourselves to periodic link scheduling schemes (similar arguments extend to any ergodic scheduling scheme as well). For fixed networks, $\mathcal{C}$ is the convex hull of the set $\{C_k\}$, which has a finite cardinality. Hence, any vector in $\mathcal{C}$ can be written as a linear combination of the C_k's. Thus, to maximize the supportable UMF, it is sufficient to optimize over transmission schemes with periodic scheduling of links where the periodic schedule corresponds to time division between the C_k's.

We obtain the following general scaling of delay.

Theorem 4: Let $S(n)$ be the total number of transmissions by the n wireless nodes on average per unit time.[6] When data is transmitted according to rate matrix $\lambda \in \Lambda$, the average delay $H(n)$, over all packets scales as

$$H(n) = \Theta\left(\frac{S(n)}{\bar{\lambda}}\right), \quad \text{where } \bar{\lambda} = \sum_{i,j} \lambda_{ij}.$$

Proof: Let Γ denote the set of all possible paths (without cycles) in the network. The amount of flow generated at node i to be transmitted to node j is λ_{ij}. Let us consider an arbitrary but fixed[7] routing scheme where a fraction α_{ij}^{γ} of the flow from node i to node j is routed over path $\gamma \in \Gamma$. We assume that the traffic matrix λ is feasible. Hence, there exists a link scheduling and routing scheme to support it. The total number of transmissions per unit time at node l is $\sum_{\gamma \ni l} \sum_{i,j} \alpha_{ij}^{\gamma} \lambda_{ij}$. Hence, the average number of transmissions per unit time in the entire network, denoted by $S(n)$, is

$$S(n) = \sum_{l=1}^{n} \sum_{\gamma \ni l} \sum_{i,j} \alpha_{ij}^{\gamma} \lambda_{ij} = \sum_{\gamma} H^{\gamma} \sum_{i,j} \alpha_{ij}^{\gamma} \lambda_{ij}$$

where H^{γ} is the number of hops on path γ. The total flow over a path γ is $\sum_{i,j} \alpha_{ij}^{\gamma} \lambda_{ij}$, i.e., the fraction of total flow over path γ is $\sum_{i,j} \alpha_{ij}^{\gamma} \lambda_{ij} / \bar{\lambda}$. Hence, the average number of hops traversed by all packets is given by

$$H(n) = \frac{1}{\bar{\lambda}} \sum_{\gamma} H^{\gamma} \sum_{i,j} \alpha_{ij}^{\gamma} \lambda_{ij} = \frac{S(n)}{\bar{\lambda}}. \qquad \square$$

We note that the above result uses very little information about the specific underlying transmission scheme. For example, consider the link scheduling scheme in the proof of Corollary 1, where we partition the set of links E into subsets $E_1, \ldots, E_\kappa$ such that all the links in each subset E_i can transmit simultaneously. Note that this scheme can support UMF $f = \Omega\left(\frac{\Phi}{\kappa n \log n}\right)$. For this transmission scheme, every link transmits at rate 1 for at most $1/\kappa$ fraction of the time. Hence, we have $S(n) \leq \frac{|E|}{\kappa}$. Thus, it follows from Theorem 4 that $H(n) = O\left(\frac{|E| \log n}{n\Phi}\right)$.

C. Computational Methods

We now describe computational methods to obtain bounds on f^* (the extensions to PMF are straightforward). As noted earlier, for wire-line networks, the computation of f^* is equivalent to solving an LP. However, in a wireless network, the link capacity is a function of the link schedule. Since, the number of link schedules is combinatorial, determining the maximum UMF is hard. Specifically, the question of checking the feasibility of a rate vector λ was proved to be NP-hard by Arikan [21]. In particular, there exists an interference model and graph

[6]The quantity $S(n)$ is well defined since we consider periodic scheduling of links.

[7]Here, we consider a fixed deterministic scheme. However, it is easy to see that the result extends for any randomized scheme as well.

under which checking the feasibility of λ is NP-hard. Motivated by this, here we address the question of providing a simple computational method to bound f^*. We use ideas of node coloring to induce a link schedule in a way similar to, for example, [22]. In particular, we can obtain an upper bound f_1^* and a lower bound f_2^* for maximum UMF f^* in polynomial time such that

$$f_1^* \leq \kappa f_2^*.$$

The upper bound can be computed by solving the LP in (1) with $C(e) = 1$ for all $e \in E$. For the lower bound, since the dual graph G^D has chromatic number κ, we can color the nodes of G^D (which are given by the set E of wireless links) such that no two nodes which share an edge share the same color. This, in turn, induces a link scheduling scheme, where each link in E is scheduled for at least a fraction $1/\kappa$ of time, and the resulting C is such that $C(e) \geq 1/\kappa$ for all $e \in E$. Again, the lower bound can be computed by solving the LP in (1) with $C(e) = 1/\kappa$ for all $e \in E$. It is easy to see that $f_1^* \leq \kappa f_2^*$.

Now from Theorem 3, we know that $\Omega\left(\frac{\Psi^*}{\log n}\right) \leq f^* \leq \Psi^*$. Thus, we can now also bound Ψ as

$$f_2^* \leq \Psi^* \leq O\left(f_1^* \log n\right).$$

Thus, the upper and lower bounds differ by at most a factor of $\kappa \log n$. In addition, using the algorithm in [18], we can find a vector $C(e)$ and the corresponding cut (U, U^C) such that the capacity of this cut

$$\frac{\sum_{(i,j): i \in U, j \in U^c} C(i,j)}{|U||U^c|}$$

is within a factor $\kappa(\log n)^2$ of Ψ.

D. Application

We now illustrate our results for the combinatorial interference model through an application to geometric random graphs. The geometric random graph has been widely used to model the topology of wireless networks after the work of Gupta and Kumar [1]. However, it has been a combinatorial object of interest for more than 60 years. We derive scaling laws for a combinatorial interference model which is more restrictive than the protocol model. Note that the lower bound hence obtained is also a lower bound for the protocol model. Specifically, the lower bound is weaker by $\log^{2.5} n$ compared to the lower bound obtained in [1]. We show that the scaling of the lower bound is closely tied to conductance of a geometric random graph.

We first define the restricted protocol interference model. It is also parameterized by the maximum radius of transmission r, and the amount of acceptable interference η.

(a) A node i can transmit to any node j if the distance between i and j, r_{ij}, is less than the transmission radius r.

(b) For transmission from node i to j to be successful, no other node k within distance $(1+\eta)r$ (where $\eta > 0$ is a constant) of node j should transmit simultaneously.

We now state a version of the well-known Chernoff bound for binomial random variables that we use multiple times in this paper.

Lemma 5: Let $X_1, \ldots, X_N$ be independent and identically distributed (i.i.d.) binary random variables with $\Pr(X_1 = 1) = p$. Let $S_n = \sum_{k=1}^n X_k$ for $n = 1, \ldots, N$. Then, for any $\delta \in (0,1)$

$$\Pr\left(|S_n - np| \geq \delta np\right) \leq 2 \exp\left(-\frac{\delta^2 np}{2}\right).$$

Specifically, for $\delta = \sqrt{\frac{2L \log n}{np}}$, we have

$$\Pr\left(|S_n - np| \geq \sqrt{2Lnp \log n}\right) \leq \frac{2}{n^L}.$$

Consider n wireless nodes distributed uniformly at random in a unit square, and the interference model given by the restricted protocol model with transmission radius r. We denote such a wireless network by $G(n,r)$. It is well known that for $G(n,r)$ to be connected with high probability, it is necessary to have $r = \Omega(\sqrt{\log n / n})$. We take $r = \Theta(\log^{3/4} n / \sqrt{n})$ and prove the following bounds on the maximum UMF, f^*, for the restrictive protocol model; the lower bound is only $\log^{2.5} n$ weaker than the result of Gupta and Kumar for the protocol model with $r = \Theta(\sqrt{\log n / n})$.

Lemma 6: For $G(n,r)$, with $r = \Theta(\log^{3/4} n / \sqrt{n})$, maximum UMF is bounded as

$$\Omega\left(\frac{1}{n^{3/2} \log^{5/2} n}\right) \leq f^* \leq O\left(\frac{1}{n^{3/2} \log^{3/4} n}\right).$$

Proof: To prove the above bounds, we obtain appropriate upper and lower bounds on the quantity Ψ^*. These bounds along with Theorem 3 imply Lemma 6. To obtain an upper bound on Ψ^*, we evaluate the cut-capacity for a specific cut-set. For the lower bound, we first, establish that a grid graph on n nodes is a subgraph of $G(n,r)$ and then use the known conductance of the grid graph.

First, consider the upper bound on Ψ^*. Specifically, consider the square, say $\mathcal{S}$, of area $1/9$ (of side $1/3$) that is in the center of the unit square. Let S be the set of nodes that fall inside this square. By definition, we have

$$\Psi^* \leq \Psi(S) = \sup_{C \in \mathcal{C}} \frac{\sum_{i \in S, j \in S^c} C(i,j)}{|S||S^c|}.$$

Therefore, it is sufficient to obtain an upper bound on $\Psi(S)$.

Corresponding to node i, define a random variable $X_i \in \{0,1\}$ which is 1 if i is in S, and 0 otherwise. Since nodes are placed uniformly and independently at random in the unit area square, X_i are i.i.d. binary random variable with $\Pr(X_i = 1) = 1/9$. Now, $\sum_{i=1}^n X_i$ is the number of nodes in S. Using Lemma 5 with $\delta = 0.009$, it follows that for large enough n, $|S| \in (0.1n, 0.2n)$ (and so $|S^c| \in [0.8n, 0.9n]$) with probability at least $1 - n^{-4}$. Now, consider squares $\mathcal{S}^0, \mathcal{S}^1$ of sides $1/3 + 2r$ and $1/3 - 2r$, respectively, with their centers being the same as that of $\mathcal{S}$. That is, $\mathcal{S}^1 \subset \mathcal{S} \subset \mathcal{S}^0$. Let $\mathcal{A}^0 = \mathcal{S}^0 - \mathcal{S}$ and $\mathcal{A}^1 = \mathcal{S} - \mathcal{S}^1$. This is illustrated in Fig. 1. Thus, $\mathcal{A}^1$ is a strip of width r surrounding $\mathcal{S}$ and $\mathcal{A}^0$ is a strip of width r on the boundary and inside $\mathcal{S}$. Since, $r = \Theta(\log^{3/4} n / \sqrt{n})$, it can be easily shown that $\mathcal{A}^0$ and $\mathcal{A}^1$ have an area of $\Theta(r)$.

Now, nodes that are in S (i.e., located inside $\mathcal{S}$) can only be connected to those nodes in S^c that lie in $\mathcal{A}^1$. Similarly, nodes in S^c that are connected to nodes S must lie in $\mathcal{A}^1$. Thus, nodes

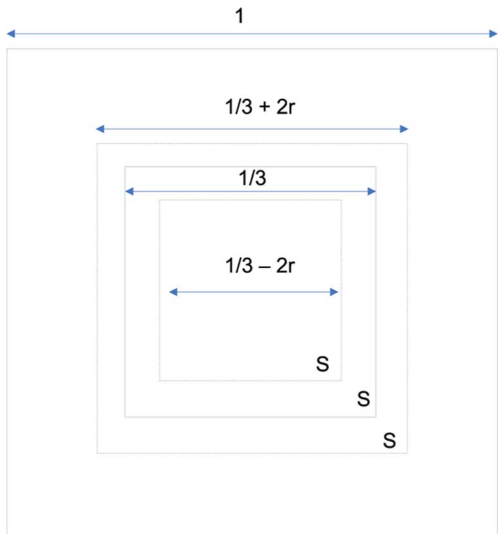

Fig. 1. Construction of the regions $\mathcal{S}$, $\mathcal{S}^0$, and $\mathcal{S}^1$.

that can communicate across the cut (S, S^c) must lie within a region of area $\Theta(r)$. For the protocol model, if a node transmits, nodes within distance $r(1+\eta)$ of the receiver must not transmit. That is, each transmission effectively silences nodes within an area of $\Theta(r^2)$. Thus, at any given time, the maximum number of simultaneous transmissions between S and S^c is $\Theta(1/r)$. This along with $|S|, |S^c| = \Theta(n)$ implies that

$$\Psi^* \le \Psi(S) = \frac{O(1/r)}{\Theta(n^2)} = O\left(\frac{1}{n^2 r}\right) = O\left(\frac{1}{n^{3/2}\log^{3/4} n}\right).$$

For the lower bound, we identify a grid subgraph of $G(n,r)$ with $r = \Theta(\log^{3/4} n/\sqrt{n})$. Consider a grid graph G_n of $\sqrt{n} \times \sqrt{n}$ nodes with each node connected to one of its four neighbors (with suitable modifications at the boundaries). The nodes of G_n are placed in a uniform manner in a unit square; each node is at a distance $1/\sqrt{n}$ from its neighbors. Now consider a minimax matching between nodes of G_n and n randomly placed nodes in the unit square, where a minimax matching is a perfect matching between the n nodes of G_n and the nodes of $G(n,r)$ with maximum length minimized. Leighton and Shor [23] established that the maximum edge length in a minimax matching, say r^*, is $\Theta(\log^{3/4} n/\sqrt{n})$ with probability at least $1 - 1/n^4$. Now we identify the subgraph G'_n (with grid graph structure) of $G(n,r)$ as follows. G'_n has all n nodes. Consider the minimax matching between G_n and $G(n,r)$. If a node of $G(n,r)$ is connected to node number m of G_n, then renumber it as m to obtain nodes of G'_n. Now by setting $r \ge r^* + 2/\sqrt{n}$, clearly a node m and m' are connected in G'_n if they are connected in G_n. Thus, we have established that $G_n \subset G'_n$. Now, we will focus only on the edges of $G(n,r)$ that belong to G'_n and provide them with positive capacity by an appropriate communication scheme that is feasible for the restricted protocol model. For this, note that in $G(n,r)$ each node is connected to at most $O(\log^{3/2} n)$ nodes with probability at least $1 - 1/n^4$ (using the Chernoff and Union bounds) for large enough n. Hence, using a simple time-division multiple-access(TDMA) scheme based on vertex coloring of $G(n,r)$, each node gets to transmit once in every $\Theta(1/\log^{3/2} n)$ time slots. This transmission can be along any outgoing edge. Since we are interested in providing positive capacity to only at most four outgoing edges, we have established that there is a simple TDMA scheme which provides $\Theta(1/\log^{3/2} n)$ capacity to each edge of a grid subgraph of $G(n,r)$. To complete the proof, we recall that the conductance of a grid graph is $\Theta(1/\sqrt{n})$[24]. That is

$$\Phi(G_n) = \min_S \frac{\sum_{i\in S, j\in S^c} \mathbf{1}_{\{(i,j)\in E\}}}{|S||S^c|} = \Theta\left(\frac{1}{n^{3/2}}\right).$$

Now, putting all the preceding discussion together we have the following:

$$\begin{aligned}\Psi^* &= \sup_{C\in\mathcal{C}} \min_{S\subset V} \frac{\sum_{i\in S, j\in S^c} C(i,j)}{|S||S^c|} \\ &\ge \Phi(G_n)\Theta\left(\frac{1}{\log^{3/2} n}\right) \\ &= \Omega\left(\frac{1}{n^{3/2}\log^{3/2} n}\right). \end{aligned} \tag{4}$$

We conclude the proof by noting that the upper and lower bounds on Ψ^* along with Theorem 3 imply Lemma 6. □

Now, we briefly discuss delay scaling. In [4], delay was defined as the average number of hops per packet, and the packet size was assumed to scale to an arbitrarily small value. For any communication scheme feasible for the protocol model with maximum transmission radius $r = \Theta(\log^{3/4} n/\sqrt{n})$, the maximum number of transmissions per unit time is upper-bounded as $O(n/\log^{3/2} n)$. Using this and Theorem 4 we obtain the following result immediately.

Corollary 2: The delay $H(n)$ for any scheme achieving $f^* = \Omega\left(\frac{1}{n^{3/2}\log^{5/2} n}\right)$ is bounded above as

$$H(n) = O\left(\sqrt{n}\log n\right).$$

IV. Gaussian Fading Channel Model

In the preceding section, we assumed that the wireless network is defined by two graphs G and G^D. We extended the results of Leighton and Rao to wireless networks modeled by a combinatorial interference model; this mainly exploited the fact that all possible transmission schemes could be described in terms of routing over a set of capacitated graphs, where the set of edge capacity vectors belonged to the convex hull of a finite number of vectors. Thus, in this sense, the inherent *discrete* nature of the model worked to our advantage.

While the combinatorial interference model can allow for arbitrary scheduling and routing schemes, it does not model all the degrees of freedom in a wireless network. Specifically, the results are not *information-theoretic*. In this section, we provide an information-theoretic characterization of the maximum PMF in a wireless network with Gaussian fading channels. The techniques for the combinatorial model can be easily extended to obtain a feasible scheme and a lower bound on the maximum PMF f^*. However, for information-theoretic upper bounds we have to work harder, especially to obtain a bound that relates to the lower bound and allows us to quantify the gap.

Our key contribution is in quantifying the suboptimality of the UMF/PMF for a *simple feasible* scheme, and an upper bound on the UMF/PMF for an arbitrary network topology, in terms of a simple graph property. The bound is general when channel side information (CSI) is assumed to be available only at the receiver (see Theorem 5 and Corollary 3). For additive white Gaussian noise (AWGN) channels, we state the result for PMF (see Theorem 6) and quantify the gap for UMF when the signal-to-noise ratio (SNR) is low enough (see Corollary 4). To the best of our knowledge, this is the first such result which guarantees that a feasible scheme achieves rates within a certain factor of an outer bound for an arbitrary network topology. We also illustrate these results through an application, in particular, by evaluating the maximum UMF for a geometric random network. This recovers many of the known results in a systematic manner.

Our main approach is as follows. We construct two directed capacitated graphs G^U and G^L for the given wireless network. The graph G^U is such that the capacity (defined appropriately later) of each cut in G^U upper-bounds the corresponding cut-capacity in the wireless network. The graph G^L is such that there exists a communication scheme that simultaneously achieves the capacity of each edge in G^L, and the ratio of capacity of each cut in G^U and G^L is bounded above by a quantifiable term. This leads to an approximate characterization of PMF in an arbitrary wireless network with Gaussian fading channels. Moreover, the feasible scheme that induces the capacities in G^L supports PMF which is within a quantifiable factor of the optimal.

A. Channel Model

This is similar to the model in, for example, [10]. We have $V = \{1, \dots, n\}$ wireless nodes with transceiver capabilities located arbitrarily in a plane. Node transmissions happen at discrete times, $t \in \mathbb{Z}_+$. Let $X_i(t)$ be the signal transmitted by node i at time $t \in \mathbb{Z}_+$. We assume that each node has a power constraint[8] such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} |X_i^2(t)| \le P.$$

Then $Y_i(t)$, the signal received by node i at time t, is given by

$$Y_i(t) = \sum_{k \neq i} H_{ik} X_k(t) + Z_i(t) \tag{5}$$

where $Z_i(t)$ denotes a complex zero-mean white Gaussian noise process with independent real and imaginary parts with variance $1/2$ such that $Z_i(t)$ are i.i.d. across all i. Let r_{ij} denote the distance between nodes i and j. Let $H_{ik}(t)$ be such that

$$H_{ik}(t) = \sqrt{g(r_{ik})} \hat{H}_{ik}(t)$$

where $\hat{H}_{ik}(t)$ is a stationary and ergodic zero mean complex Gaussian process with independent real and imaginary parts and variance $1/2$, i.e., we assume $\hat{H}_{ik}(t)$'s to be circularly symmetric Gaussian random variables. This models channel fluctuations due to frequency flat fading. Also, $g(\cdot)$ is a monotonically decreasing function that models path loss. It satisfies $g(x) \le 1$ for all $x \ge 0$. We also assume that the $\hat{H}_{ik}(t)$'s are independent.

[8]For notational simplicity we assume that each node has the same power constraint. The general case, where each node has different maximum average power can be handled using identical techniques.

B. Graph Definitions

Consider the following two graphs induced by a wireless network of n nodes:

(1) K_n is the fully connected graph with node set V;
(2) G_r is the graph where each node $i \in V$ is connected to all nodes that are within a distance r of i. Let E_r denote the edge set of G_r. Let $\Delta(r)$ be the maximum vertex degree of G_r. Finally, define

$$r^* = \min\{r : G_r \text{ is connected}\}.$$

C. Preliminaries

We utilize the following two simple lemmas in the analysis in this section.

Lemma 7: Given $x_i \in (0,1), 1 \le i \le N$

$$\sum_{i=1}^{N} \log(1+\sqrt{x_i}) \le \sqrt{2N} \sqrt{\sum_{i=1}^{N} \log(1+x_i)}.$$

Proof: See Appendix B. □

Lemma 8: For any $x \ge 0$, $\alpha \in (0,1)$, $\frac{1}{\alpha}\log(1+\alpha x) \ge \log(1+x)$.

Proof: See Appendix B. □

D. Results

We obtain bounds on the maximum PMF for three different cases:

(1) fading channel with AWGN, and channel side information (CSI) available only at the receiver (Theorem 5 and Corollary 3);
(2) deterministic (no fading) AWGN channel (Theorem 6 and Corollary 4); and
(3) fading channel with AWGN, and CSI available at both the transmitter and the receiver (Theorem 7).

The exact bounds for the above cases are different, but the analysis and bounding techniques are similar.

1) Random Fading With Rx-Only CSI: We first obtain bounds on the maximum supportable PMF for Gaussian channels with random fading under the assumption that CSI is available at the receiver, but not the transmitter. We then relate the bounds for PMF, and show that the gap can be quantified well, and under very general assumptions. We note that this is the case for which we can obtain the strongest results.

Theorem 5: With CSI available only at the receivers, f_π^* is bounded as

$$f_\pi^* \le \min_{S \subset V} \frac{\sum_{i\in S, j\in S^c} \mathbb{E}\left(\log\left(1+P|H_{ji}|^2\right)\right)}{\pi(S)\pi(S^c)}$$

$$f_\pi^* = \Omega\left(\sup_{r \ge r^*,\, \eta \ge 0} \left[\frac{1}{1+\Delta(r)\Delta(r(1+\eta))}\right] \times \left[\min_{S\subset V} \frac{\sum_{i\in S, j\in S^c} \mathbf{1}_{(i,j)\in E_r} \mathbb{E}\log\left(1+\frac{P|H_{ji}|^2}{1+nPg(r(1+\eta))}\right)}{\log p_\pi \pi(S)\pi(S^c)}\right]\right).$$

Theorem 5 provides bounds on f_π^* which relate to the "cut capacity" of appropriate capacitated graphs. Specifically, we can compute the information-theoretic bounds (for any PMF) in polynomial time using flow arguments, and by solving an LP as detailed in Section IV-E. However, it is not clear how *tight* these bounds are. We now quantify the *gap* between the upper and lower bounds.

Corollary 3: For any $r \geq r^*$, denote

$$\delta(r) = \max_i \sum_{j: r_{ij} \geq r} Pg(r_{ij}).$$

Then

$$\Omega\left(\frac{\Psi}{(1+\Delta^2(r))(1+\delta(r))\log p_\pi}\right) \leq f_\pi^* \leq (1+\gamma(r))\Psi$$

where

$$\Psi = \min_{S \subset V} \frac{\sum_{i \in S, j \in S^c: r_{ij} \leq r} \mathbb{E}\log(1+P|H_{ij}|^2)}{\pi(S)\pi(S^C)}$$

and

$$\gamma(r) = \max_{S \subset V: \pi(S), \pi(S^C) > 0} \frac{\sum_{i \in S, j \in S^c: r_{ij} > r} \mathbb{E}\log(1+P|H_{ij}|^2)}{\sum_{i \in S, j \in S^c: r_{ij} \leq r} \mathbb{E}\log(1+P|H_{ij}|^2)}.$$

Note that both $\gamma(r)$ and $\delta(r)$ are decreasing functions of r, and $\Delta(r)$ is an increasing function of r. Also, since power typically decays as $1/r^a$ for $2 \leq a \leq 6$, while for uniformly distributed networks $\Delta(r)$ grows only linearly with r, the decay of $\gamma(r)$ and $\delta(r)$ is much faster than the growth of $\Delta(r)$. Hence, for r large enough, the *gap* is dominated by the term $\log p_\pi(1+\Delta(r)^2)$. Specifically, assume that there exists an $\epsilon > 0$ such that the graph $\hat{G}^\epsilon = (V, \hat{E}^\epsilon)$ is connected, where $\hat{E}^\epsilon = \{(i,j) : \mathbb{E}\left[\log\left(1+P|H_{ij}|^2\right)\right] \geq n^{-\epsilon/2}\}$. Then the above bound for UMF reduces to [25]

$$\Omega\left(\frac{\text{min-cut}_R}{\Delta^2(r_\epsilon)\log n}\right) = f^* = O(\text{min-cut}_R)$$

where

$$\text{min-cut}_R = \min_{S \subset V} \frac{\sum_{i \in S, j \in S^c: r_{ij} \leq r_\epsilon} \mathbb{E}\log(1+P|H_{ij}|^2)}{|S||S^C|}$$

and r_ϵ is such that $\delta(r_\epsilon) \leq \frac{1}{n^{1+\epsilon}}$.

We provide the proofs of Theorem 5 and Corollary 3 below. The main idea in the proof of Theorem 5 is to neglect interference to upper-bound-achievable rates on links, and to construct a transmission scheme to induce achievable rates on the links. In particular, the scheme that we construct consists of time sharing between multiple transmission schemes, each of which enables direct transmissions between nodes that are separated by at most distance r. Then the lower bound on f^* is obtained by routing over graph G_r, where each edge has a capacity given by this time division scheme.

Proof: [Theorem 5] We first prove the upper bound. Following the steps in the proof of Theorem 2.1 in [10] and using $(1+\sum_{i=1}^n \alpha_i) \leq \prod_{i=1}^n (1+\alpha_i)$ for $\alpha_i > 0$, we obtain for $\lambda \in \Lambda$

$$\begin{aligned}\sum_{i \in S, j \in S^c} \lambda_{ij} &\leq \max_{Q_S \succeq 0, (Q_S)_{ii} \leq P} \mathbb{E}\left[\log\det\left(I + H_S Q_S H_S^*\right)\right] \\ &\leq \sum_{i \in S, j \in S^c} \mathbb{E}\left(\log(1+P|H_{ji}|^2)\right). \quad (6)\end{aligned}$$

Now, for any PMF $M = M(f_\pi, \pi)$, it must be that

$$\sum_{i \in S, j \in S^C} M_{ij} = f_\pi \pi(S)\pi(S^C).$$

Hence, for any such PMF $M(f,\pi) \in \Lambda$, the upper bound in the theorem holds.

To establish the lower bound, we construct a transmission scheme for which the PMF is greater than or equal to that in the lower bound. For $r \geq r^*$, consider the graph $G_r = (V, E_r)$ on the n nodes defined above. We use $\Delta(r(1+\eta))$ to denote the maximum vertex degree of the graph $G_{r(1+\eta)}$. Now, consider the following transmission scheme. A node i can transmit to a node j only if $r_{ij} \leq r$. Also, when a node i transmits, no node within a distance $r(1+\eta)$ of the receiver can transmit. This is illustrated in Fig. 2. Thus, when a link $(i,j) \in E_r$ is active, at most $\Delta(r(1+\eta))$ nodes are constrained to remain silent, i.e., at most $\alpha = \Delta(r(1+\eta))\Delta(r)$ links are constrained to remain inactive. Hence, the chromatic number of the dual graph is at most $(1+\Delta(r(1+\eta))\Delta(r)$. In addition, we assume that the signal transmitted by each node has a Gaussian distribution. For any given link that transmits data at a particular time, we treat all other simultaneous transmissions in the network as interference. Now focus on any one link, say link $(1,2)$ between node 1 and 2, without loss of generality. We claim the following.

Lemma 9: For the above scheme, the following rate on link $(1,2)$ is achievable

$$\lambda_{12} = \alpha^{-1}\mathbb{E}\log\left(1 + \frac{P|H_{21}|^2}{1+nPg(r(1+\eta))}\right).$$

We prove Lemma 9 in Appendix B. We now explain how it implies the proof of Theorem 5. A similar analysis holds for other links in E_r as well. Thus, for graph G_r the following rate is jointly achievable on each link $(i,j) \in E_r$:

$$\alpha^{-1}\mathbb{E}\log\left(1 + \frac{P|H_{ji}|^2}{1+nPg(r(1+\eta))}\right).$$

Now given the capacitated graph G_r, we can use routing over (see Section II-B) to obtain a PMF that is lower-bounded by the following quantity:

$$f_{\text{LB}}(r,\eta) = \Omega\left(\left[\frac{1}{1+\Delta(r)\Delta(r(1+\eta))}\right] \times \left[\min_{S \subset V} \frac{\sum_{i \in S, j \in S^c} \mathbf{1}_{(i,j) \in E_r} \mathbb{E}\log\left(1 + \frac{P|H_{ji}|^2}{1+nPg(r(1+\eta))}\right)}{\log p_\pi \pi(S)\pi(S^c)}\right]\right).$$

This implies the following lower bound on f_π^*:

$$f_\pi^* \geq \sup_{r \geq r^*, \eta \geq 0} f_{\text{LB}}(r,\eta).$$

This is precisely the lower bound in the statement of Theorem 5. □

Proof: [Corollary 3] Consider any S such that $\pi(S)$, $\pi(S^C) > 0$. Then

$$\begin{aligned}\text{Cut}(S, S^c) &\triangleq \sum_{i \in S, j \in S^c} \mathbb{E}\left(\log(1+P|H_{ji}|^2)\right) \\ &\leq (1+\gamma(r)) \sum_{i \in S, j \in S^c: r_{ij} \leq r} \mathbb{E}\left(\log(1+P|H_{ji}|^2)\right) \quad (7)\end{aligned}$$

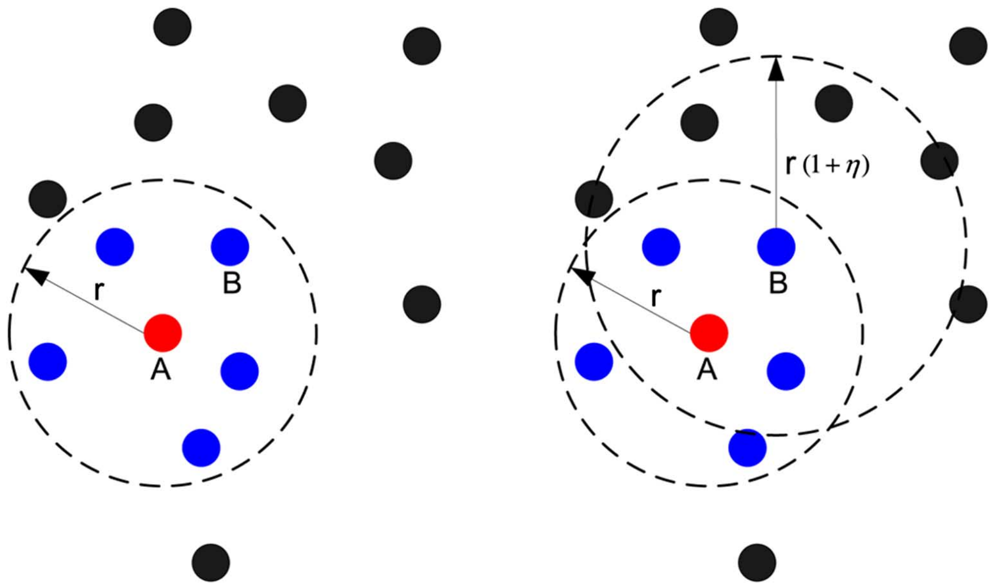

Fig. 2. Transmission scheme for lower bound in case of Rx-only CSI. *Left.* A node can transmit to any other node within distance r. *Right.* No node within distance $r(1+\eta)$ of the receiver (node B) can transmit at the time when node A transmits to node B.

where the second line follows from the concavity of the log function, Jensen's inequality, $\log(1+x) \le x$ for $x > 0$, and definition of $\gamma(r)$. Thus

$$\frac{\mathsf{Cut}(S,S^c)}{\pi(S)\pi(S^C)} \le \Psi(1+\gamma(r)). \tag{8}$$

The upper bound then follows from the upper bound in Theorem 5.

Next, we consider the transmission scheme that led to the lower bound in (23) with $\eta = 0$. Note that in (23), we used the term $nPg(r(1+\eta))$ as a bound on the interference power. However, here we consider the actual interference $I_{ij} = \sum_{k\in V: r_{jk}\ge r} Pg(r_{jk})$ for a transmission from i to j. Note that $I_{ij} \le \delta(r)$. Now, by Lemma 8, we have

$$\mathbb{E}\left[\log\left(1+\frac{P|H_{ji}|^2}{1+I}\right)\right] \ge \frac{1}{1+\delta(r)}\mathbb{E}\left(\log\left(1+P|H_{ji}|^2\right)\right). \tag{9}$$

Using (10) and (12) along with the lower bound obtained via the time-division scheme that led to (23), the lower bound in Theorem 5 gives us

$$\begin{aligned} f^* &= \Omega\left(\min_{S\subset V}\frac{\sum_{i\in S, j\in S^c: r_{ij}\le r}\mathbb{E}\left(\log(1+P|H_{ji}|^2)\right)}{\pi(S)\pi(S^C)(1+\Delta(r)^2)(1+\delta(r))\log p_\pi}\right)\\ &= \Omega\left(\frac{\Psi}{(1+\Delta(r)^2)(1+\delta(r))\log p_\pi}\right). \end{aligned} \tag{10}$$

□

2) AWGN Channels: We now consider an AWGN channel without fading, i.e., we have $\hat{H}_{kj} = 1$ w.p. $1, \forall\, k,j = 1,\ldots,n$. We first obtain the following set of bounds on maximum PMF using standard arguments.

Theorem 6: The maximum PMF f_π^* is bounded as

$$\begin{aligned} f_\pi^* &\le \min_{S\subset V}\frac{2\sum_{i\in S, j\in S^c}\log(1+\sqrt{Pg(r_{ij})})}{\pi(S)\pi(S^c)},\\ f_\pi^* &= \Omega\left(\sup_{r\ge r^*,\,\eta\ge 0}\left[\frac{1}{1+\Delta(r)\Delta(r(1+\eta))}\right]\right.\\ &\quad\left.\times\left[\min_{S\subset V}\frac{\sum_{i\in S, j\in S^c: r_{ij}\le r}\log\left(1+\frac{Pg(r_{ij})}{1+nPg(r(1+\eta))}\right)}{\log p_\pi \pi(S)\pi(S^c)}\right]\right). \end{aligned}$$

Next, we present a corollary of Theorem 6 which characterizes the tightness of the above bound for UMF for low enough SNR.

Corollary 4: Define

$$I(r) = \min\{I > 0 : \sum_{j: r_{ij}\ge r} Pg(r_{ij}) \le I,\ \text{for all } i\}$$

and

$$r(\delta) = \min\{r > 0 : I(r) \le \delta\}, \quad \delta > 0.$$

Then, for $P < 1$

$$\Omega\left(\frac{n}{(1+\delta)\Delta(r(\delta))(1+\Delta(r(\delta))^2)\log n}\Psi^2\right) \le f^* \le 2\Psi + O\left(\frac{\delta}{n}\right)$$

where

$$\Psi = \min_{U\subset V}\frac{\sum_{i\in U, j\in U^c: r_{ij}\le r(\delta)}\log\left(1+\sqrt{Pg(r_{ij})}\right)}{|U||U^C|}.$$

We now present the proofs of Theorem 6 and Corollary 4.

Proof: [Theorem 6] We first prove the upper bound. In order to bound the sum-rate across each given cut, we refer to the proof of the max-flow min-cut lemma in [9], which yields for any $S \subset V$ and $\lambda \in \Lambda$

$$\sum_{i \in S, j \in S^c} \lambda_{ij} \leq \sum_{j \in S^c} \log\left(1 + \mathbb{E}\left(|\tilde{X}_j|^2\right)\right)$$

where $\tilde{X}_j = \sum_{i \in S} \sqrt{g(r_{ji})}\, X_i$. We therefore deduce that

$$\begin{aligned}\sum_{i \in S, j \in S^c} \lambda_{ij} &\leq \sum_{j \in S^c} \log\left[1 + \sum_{i,k \in S} \sqrt{g(r_{ji})\, g(r_{jk})}\, |\mathbb{E}(X_i \overline{X_k})|\right] \\ &\leq \sum_{j \in S^c} \log\left[1 + P\left(\sum_{i \in S} \sqrt{g(r_{ji})}\right)^2\right]\end{aligned}$$

since $|\mathbb{E}(X_i \overline{X_k})| \leq \sqrt{P_i P_k} \leq P$. Finally, we obtain

$$\begin{aligned}\sum_{i \in S, j \in S^c} \lambda_{ij} &\leq \sum_{j \in S^c} 2 \log\left(1 + \sqrt{P} \sum_{i \in S} \sqrt{g(r_{ji})}\right) \\ &\leq \sum_{i \in S, j \in S^c} 2 \log\left(1 + \sqrt{P g(r_{ji})}\right).\end{aligned}$$

Now, for any PMF $M = M(f_\pi, \pi)$, it must be that

$$\sum_{i \in S, j \in S^C} M_{ij} = f_\pi \pi(S) \pi(S^C).$$

Hence, for any such PMF $M(f, \pi) \in \Lambda$, the upper bound in the theorem holds.

To establish the lower bound, we construct a transmission scheme for which the PMF is greater than or equal to that in the lower bound. For $r \geq r^*$, consider the graph $G_r = (V, E_r)$ on the n nodes defined above. We use $\Delta(r(1+\eta))$ to denote the maximum vertex degree of the graph $G_{r(1+\eta)}$. Now, consider the following transmission scheme. A node i can transmit to a node j only if $r_{ij} \leq r$. Also, when a node i transmits, no node within a distance $r(1+\eta)$ of the receiver can transmit. Thus, when a link $(i,j) \in E_r$ is active, at most $\Delta(r(1+\eta))$ nodes are constrained to remain silent, i.e., at most $\Delta(r(1+\eta))\Delta(r)$ links are constrained to remain inactive. Hence, the chromatic number of the dual graph is at most $(1 + \Delta(r(1+\eta))\Delta(r)$. In addition, we assume that the signal transmitted by each node has a Gaussian distribution. Then, subject to the maximum average power constraint, for any node pair i, j, such that $r_{ij} \leq r$, the following rate is achievable from $i \to j$:

$$\lambda_{ij} \geq \frac{\log\left(1 + \frac{P g(r_{ij})}{1 + n P g(r(1+\eta))}\right)}{1 + \Delta(r)\Delta(r(1+\eta))}. \tag{11}$$

Note that the interference is due to at most n nodes and all the interfering nodes are at least a distance $r(1+\eta)$ away from the receiver. We now consider routing over the graph G_r, where each edge (i,j) has capacity λ_{ij}. The lower bound then follows from the lower bound in Theorem 2. □

Proof: [Corollary 4] Consider any cut defined by (S, S^c). Due to the symmetry of the upper bound in Theorem 6, without loss of generality, assume $|S| \leq n/2$. Consider any δ such that $r(\delta) \geq r^*$. Then

$$\begin{aligned}\mathsf{Cut}(S, S^c) &= \sum_{i \in S, j \in S^c} \log(1 + \sqrt{P g(r_{ij})}) \\ &= \sum_{i \in S, j \in S^c : r_{ij} \leq r(\delta)} \log\left(1 + \sqrt{P g(r_{ij})}\right) \\ &\quad + \sum_{i \in S, j \in S^c : r_{ij} > r(\delta)} \log\left(1 + \sqrt{P g(r_{ij})}\right) \\ &\leq \sum_{i \in S, j \in S^c : r_{ij} \leq r(\delta)} \log\left(1 + \sqrt{P g(r_{ij})}\right) + |S|\delta\end{aligned} \tag{12}$$

where the last step follows from the definition of $r(\delta)$, and $\log(1 + \sqrt{x}) \leq x$ for $x \leq 1$. Hence, the upper bound in the corollary follows from the upper bound in Theorem 6. Since we assume $P g(r_{ij}) \leq 1$ for all i and j, from Lemma 7, we have

$$\begin{aligned}&\sum_{i \in S, j \in S^c : r_{ij} \leq r(\delta)} \log\left[1 + \sqrt{P g(r_{ij})}\right] \\ &\leq \sqrt{2\Delta(r(\delta))|S| \sum_{i \in S, j \in S^c : r_{ij} \leq r(\delta)} \log\left(1 + P g(r_{ij})\right)}.\end{aligned} \tag{13}$$

For the lower bound, consider the choice of $r = r(\delta)$ and $\eta = 0$ for the scheme described in the proof of Theorem 6. Then, the interference during data transmission from i to j, $I_{ij} = \sum_{k \in V : r_{jk} \geq r(\delta)} P g(r_{jk}) \leq \delta$. Now, Lemma 8 implies that

$$\log\left(1 + \frac{P g(r_{ij})}{1 + I_{ij}}\right) \geq \frac{1}{1+\delta} \log(1 + P g(r_{ij})). \tag{14}$$

Using an appropriately modified lower bound in Theorem 6 for the choice of $r = r(\delta)$, $\eta = 0$, it follows that

$$\begin{aligned}f^* &= \Omega\left[\min_{S \subset V} \frac{\sum_{i \in S, j \in S^c : r_{ij} \leq r(\delta)} \log(1 + P g(r_{ij}))}{(1+\delta)(1 + \Delta(r(\delta))^2) \log n |S||S^c|}\right] \\ &= \Omega\left[\frac{\Psi^2 n}{(1+\delta)\Delta(r(\delta))(1 + \Delta(r(\delta))^2) \log n}\right]\end{aligned} \tag{15}$$

where the second step follows from (13). The lower bound in Theorem 6 then implies the lower bound in the corollary. This completes the proof. □

3) Random Fading With CSI at Both Tx and Rx: We now obtain bounds on the PMF for a Gaussian channel with random fading when CSI is available at both the transmitter and the receiver. Qualitatively, these bounds are very similar to the case of deterministic AWGN channels. The main result is as follows.

Theorem 7: With CSI at both transmitters and receivers, f_π^* is bounded as follows:

$$f_\pi^* \leq \min_{S \subset V} \frac{\sum_{i \in S, j \in S^c} 2\mathbb{E} \log(1 + \sqrt{P}|H_{ji}|)}{\pi(S)\pi(S^c)}.$$

The lower bound for the receiver only CSI case is a (weak) lower bound for this case as well.

Proof: The upper bound follows again from the proof of Theorem 2.1 in [10], from which we deduce that for any $\lambda \in \Lambda$

$$\sum_{i\in S, j\in S^c} \lambda_{ij} \le \mathbb{E}\left[\max_{Q\succeq 0, Q_{ii}\le P} \log\det\left(I + H_S Q_S H_S^*\right)\right] \le \sum_{j\in S^c} \mathbb{E}\left[\max_{Q\succeq 0, Q_{ii}\le P} \log\left(1 + h_j Q_S h_j^*\right)\right]$$

where h_j is the jth row of H. Since $h_j Q_S h_j^*$ is maximum when $(Q_S)_{ik} \equiv P$ for all $i,k \in S$, we obtain, following the steps of the proof of Theorem 6

$$\sum_{i\in S, j\in S^c} \lambda_{ij} \le \sum_{j\in S^c} \mathbb{E}\log\left(1 + P\left(\sum_{i\in S}|H_{ij}|\right)^2\right] \le \sum_{j\in S^c} 2\mathbb{E}\log\left(1+\sqrt{P}\left(\sum_{i\in S}|H_{ij}|\right)\right] \le \sum_{i\in S, j\in S^c} 2\mathbb{E}\left(\log(1+\sqrt{P}|H_{ij}|\right),$$

so the upper bound on f_π^* follows from Theorem 2. □

E. Computational Methods

We discuss the implications of the bounds for the case of CSI availability at only the receiver as stated in Corollary 3. Similar implications follow for the case where CSI is available to both transmitters and receivers as well.

Corollary 3 shows that an upper bound on f_π^* can be obtained via the maximum PMF on graph K_n, where each edge (i,j) has a capacity $\log\left(1 + P|H_{ij}|^2\right)$, and there is no interference; specifically, $\log n$ times the PMF thus computed for K_n is an upper bound on f^*. The lower bound is obtained via routing on G_r with edge (i,j) having capacity $\frac{\log\left(1+P|H_{ij}|^2\right)}{(1+\Delta^2(r))(1+\delta)}$. Hence, the PMF on $G_{r(\delta)}$ is a lower bound on f_π^*. Both the above computations can be done by solving an LP in polynomial time. Moreover, the ratio of the bounds is quantified in Corollary 3. We note that such an approximation ratio could be obtained easily for the combinatorial interference model using node coloring arguments. The arguments here are more complicated, as detailed in the proof of Corollary 3.

F. Application

We now apply the information-theoretic characterization of PMF in the previous subsection to obtain a scaling law for average UMF in a geometric random network with fading channels, and when CSI is available at the receivers. The scaling law we obtain is along similar lines to those that exist in the literature. However, it illustrates that using very general methods, we can obtain upper and lower bounds which are tight. Similar bounds can be obtained when CSI is available both at the transmitter and the receiver or when the channels are AWGN channels.

We consider a geometric random graph model with a constant node density: n nodes are placed uniformly at random in a torus of area n (note that this is different from the standard model of torus of unit area). Thus, the distance between two nodes is a random variable taking values in $(0, \Theta(\sqrt{n}))$. We assume that all nodes have the same transmission power equal to 1, i.e., $P_i = 1$ for all $1 \le i \le n$. We have the following result characterizing f^*.

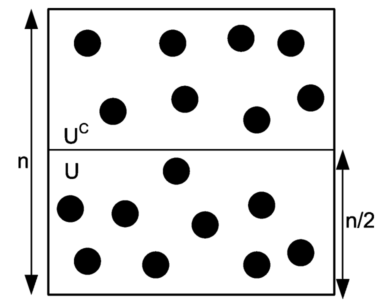

Fig. 3. Cut for obtaining upper bound.

Lemma 10: Consider the Gaussian channel model with random fading and CSI available only at the receivers. Let $g(r) = (1+r)^{-\alpha}$, $\alpha > 3$, and $P = 1$. Then for a geometric random graph with constant node density (described above), the average (over random position of nodes) f^* is bounded as

$$\Omega\left(\frac{1}{n^{3/2}\log^{1+\alpha} n}\right) \le \mathbb{E}[f^*] \le O\left(\frac{1}{n^{3/2}}\right)$$

if $\Pr(|\hat{H}_{ij}|^2 \ge \beta) \ge \gamma$ for some strictily positive constants β, γ (independent of n) for all $1 \le i,j \le n$. (Note that the condition is for the normalized channel gains $\hat{H}_{ij}$'s and not the actual gains H_{ij}'s.)

Proof: We use Theorem 5 to evaluate the bounds. First, we obtain an upper bound by evaluating the bound of Theorem 5 for a specific cut (U, U^c). Then, we evaluate lower bound by relating it to an appropriate grid-graph as in Lemma 6.

Now, we consider the upper bound. Consider a horizontal line dividing the square of area n into equal halves. Let U be set of nodes that lie in bottom half, and so U^c is the set of nodes that lie in the top half, as shown in Fig. 3. From Theorem 5, we have

$$\begin{aligned} f^*|U||U^c| &\le \sum_{i\in U, j\in U^c} \mathbb{E}\log(1+P|H_{ij}|^2) \\ &\le \sum_{i\in U, j\in U^c} \log(1+P\mathbb{E}[|H_{ij}|^2]) \\ &= \sum_{i\in U, j\in U^c} \log(1+Pg(r_{ij})) \\ &\le \sum_{i\in U, j\in U^c} Pg(r_{ij}) \\ &= \sum_{i\neq j}(1+r_{ij})^{-\alpha}\mathbf{1}_{\{i\in U\}}\mathbf{1}_{\{j\in U^c\}} \end{aligned} \tag{16}$$

where we have used Jensen's inequality $\log(1+x) \le x$ for all $x \ge 0$, and the hypothesis of the lemma. Since, the nodes are thrown uniformly at random, the expectation of each term in (16) for a pair (i,j) is the same. Using linearity of expectation, we obtain that

$$\mathbb{E}\left[\sum_{i\neq j}(1+r_{ij})^{-\alpha}\mathbf{1}_{\{i\in U\}}\mathbf{1}_{\{j\in U^c\}}\right] = n(n-1)\mathbb{E}\left[(1+r_{12})^{-\alpha}\mathbf{1}_{\{1\in U\}}\mathbf{1}_{\{2\in U^c\}}\right]$$

$$\begin{aligned}
&\leq O\left(n^2 \int_1^{\sqrt{n}} \int_r^{\sqrt{n}} s^{-\alpha} \frac{sds}{n} \frac{\sqrt{n}dr}{n}\right) \\
&= O\left(\sqrt{n} \int_1^{\sqrt{n}} \int_r^{\sqrt{n}} s^{-\alpha+1} dsdr\right) \\
&= O\left(\sqrt{n} \int_1^{\sqrt{n}} r^{-\alpha+2} dr\right) \\
&= O(\sqrt{n}) \qquad (17)
\end{aligned}$$

where we used the fact that for $\alpha > 3$ the last integral is bounded above by a constant. The above evaluation can be justified as follows. First note that $\Pr(1 \in U, 2 \in U^c) = 1/4$. Given $\{1 \in U, 2 \in U^c\}$, node 1 in the bottom rectangle and node 2 in the top rectangle are uniformly distributed. Now, consider a thin horizontal strip of width dr and length $\sqrt{n}$ at distance r below the horizontal line dividing the square (and inducing U, U^c). The node $1 \in U$ belongs to this strip with probability $2\sqrt{n}dr/n$. Now, node 2 is at distance at least r from node 1. Consider a ring of width ds, centered at node 1's position and of radius $s \geq r$. The area of this ring is $2\pi sds$. The probability of node 2 being in this ring is bounded above by $4\pi sds/n$. When the above described condition is true, nodes 1 and 2 are at distance s. Integrating over the appropriate ranges justifies the final outcome (21).

Now, it is easy to see that under any configuration of nodes, $f^* = O(n^2)$ since $g(r) \leq 1$ for any $r \geq 0$, $P = 1$, and elementary arguments. Let event $A = \{|U||U^c| = \Theta(n^2)\}$. Using Chernoff bound, it is easy to see that (with appropriate selection of constants in definition of A) for large enough n, we have

$$\Pr(A) = 1 - 1/n^6.$$

Using this estimate and bound $f^* = O(n^2)$ we obtain that

$$\begin{aligned}
\mathbb{E}[f^*] &= \mathbb{E}[f^* \mathbf{1}_A] + \mathbb{E}[f^* \mathbf{1}_{A^c}] \\
&\leq \mathbb{E}[f^* \mathbf{1}_A] + O\left(\frac{1}{n^4}\right) \\
&= \Theta\left(\frac{\mathbb{E}[f^*|U||U^c| \mathbf{1}_A]}{n^2}\right) + O\left(\frac{1}{n^4}\right) \\
&\leq \Theta\left(\frac{\mathbb{E}[f^*|U||U^c|]}{n^2}\right) + O\left(\frac{1}{n^4}\right) \\
&= O\left(\frac{1}{n^{3/2}}\right). \qquad (18)
\end{aligned}$$

Next, we prove the lower bound. For this we construct a graph with achievable link capacities for which the average f^* is lower-bounded as claimed in the lemma. Consider $r = \Theta(\log n)$. Then the corresponding G_r, which is the geometric random graph $G(n, r)$, is connected with high probability (at least $1 - 1/n^4$ by appropriate choice of constants in selection of r). For this choice of r, using the Chernoff and Union bounds it follows that with probability at least $1 - 1/n^4$

$$\Delta(2r) = \Theta\left(\log^2 n\right).$$

Again, we can identify a grid graph structure as a subgraph structure of G_r based on the argument used in Lemma 6. Denote the edges of this grid subgraph structure as $\hat{E}$. We note that $\Theta(1)$ edges are incident on each of the n nodes that belong to $\hat{E}$ (which is a property of the grid-graph structure). Next, we design a feasible transmission scheme for which each edge in $\hat{E}$ can support a transmission rate of $\Omega(\log^{-\alpha} n)$.

Specifically, we consider a TDMA schedule for the graph G_r similar to that described in the proof of the lower bound for Theorem 5. It is easy to see that G_{2r} can be vertex colored using $\Theta(\Delta(2r))$ colors. We use a randomized scheme to do TDMA scheduling as follows: in each time-slot, each node becomes tentatively active with probability $1/\Delta(2r)$ and remains inactive otherwise. If a node becomes tentatively active and none of its neighbors in G_{2r} is tentatively active, then it will become active. Else, it becomes inactive. All active nodes transmit in the time-slot simultaneously. It is easy to see that each node transmits for $\Theta(1/\Delta(2r))$ fraction of the time on an average. The randomization here is used to facilitate the computation of a simple bound on the average interference experienced by a node due to transmissions by nodes that are not its neighbor.

Now under the above vertex coloring, each node gets to transmit once in $\Theta(\Delta(2r))$ time-slots on average at power $\Theta(P\Delta(2r)) = \Theta(\Delta(2r))$. We wish to concentrate on transmissions for edges that belong to $\hat{E}$, which are a subset of edges of G_r. For any such transmission, say from $u \to v$ with $(u, v) \in \hat{E}$, according to the above coloring of G_{2r}, no other node within distance r of v transmits simultaneously. Also, any node that is at least a distance r away from v can be active with probability at most $1/\Delta(2r)$. Hence, the average power corresponding to the interference received by node v, say I_v, can be bounded above as follows:

$$\begin{aligned}
I_v &= O\left(\sum_{j: r_{vj} > r} \frac{P\Delta(2r)\mathbb{E}\left[|H_{ij}|^2\right]}{\Delta(2r)}\right) \\
&= O\left(\sum_{j: r_{vj} > r} g(r_{vj})\right). \qquad (19)
\end{aligned}$$

where we used the fact that each node transmits at power $P\Delta(2r) = \Delta(2r)$ for $1/\Delta(2r)$ fraction of the time and $\mathbb{E}\left[|H_{ij}|^2\right] = g(r_{ij})$. By another application of Chernoff and Union bounds, it can be shown that the number of nodes in an annulus around node v with unit width and radius R for $R \in \mathbb{N}$, $r \leq R \leq \sqrt{n}$, is $\Theta(R)$ with probability at least $1 - 1/n^5$. Then it follows that

$$\sum_{j: r_{vj} > r} g(r_{vj}) = O\left(\sum_{R=\lfloor r \rfloor}^{\sqrt{n}} g(R)R\right) = O\left(r^{-\alpha+2}\right)$$

where we have used fact that $\alpha > 3$. Using this in (19), we obtain that with probability at least $1 - 1/n^{5/2}$ we have

$$I_v = O\left(r^{-\alpha+2}\right) = O\left(\log^{2-\alpha} n\right). \qquad (20)$$

That is, $I_v \to 0$ as $n \to \infty$ for $\alpha > 3$. Thus, by selection of large enough n, I_v can be made as small as possible. That is, when transmission from $u \to v$ happens, the average noise received by node v due to other simultaneous transmission is very small, say less than δ for some small enough $\delta > 0$.

Given this, the arguments used in Lemma 9 imply that when u transmits to v at power $P\Delta(2r)$ once in $\Theta(\Delta(2r))$ time-slot, considering other transmissions as noise, we obtain that the effective rate between $u \to v$ is lower-bounded as

$$\lambda_{u\to v} = \Omega\left(\Delta(2r)^{-1}\mathbb{E}\left[\log\left(1+\frac{P\Delta(2r)g(r_{uv})|\hat{H}_{uv}|^2}{1+I_v}\right)\right]\right).$$

Now, $\hat{H}_{uv}$ is independent of everything else and

$$\Pr(|\hat{H}_{uv}|^2 \geq \beta) \geq \gamma$$

for some positive constants β, γ as per our hypothesis. Therefore, use of Lemma 8 implies

$$\lambda_{u\to v} \geq \Omega\left(\Delta(2r)^{-1}\log\left(1+\frac{P\Delta(2r)g(r_{uv})}{1+I_v}\right)\right).$$

Further, $I_v \leq \delta$ for small enough δ. Therefore, another use of Lemma 8 implies that

$$\lambda_{u\to v} \geq \Omega\left(\Delta(2r)^{-1}\log\left(1+P\Delta(2r)g(r_{uv})\right)\right).$$

Now

$$Pg(r_{uv})\Delta(2r) = \Omega\left(\log^{2-\alpha} n\right)$$

where we have used the fact that $r_{uv} \leq r$ and $g(\cdot)$ is monotonically decreasing. For $x \in (0, 0.5)$, $\log(1+x) \geq x/2$. Therefore, for $\alpha > 3$

$$\lambda_{u\to v} = \Omega\left(\Delta(2r)^{-1}\log^{2-\alpha} n\right).$$

Since $\Delta(2r) = \Theta(\log^2 n)$, we have established that the effective capacity of transmissions for each edge under the above described TDMA scheme is $\Omega(\log^{-\alpha} n)$. That is, each edge of $\hat{E}$ gets capacity at least $\Omega(\log^{-\alpha} n)$. Now recall that a grid graph with unit capacity has f^* lower-bounded as $\Omega\left(\frac{1}{n^{3/2}\log n}\right)$. Hence, using this routing of UMF along edges of $\hat{E}$ with capacity $\Omega(\log^{-\alpha} n)$ we obtain that

$$f^* = \Omega\left(\frac{1}{n^{3/2}\log^{1+\alpha} n}\right).$$

By careful accounting of probability of relevant events above and the Union bound of events will imply that the above stated lower bound on f^* holds with probability at least $1-1/n^2$. Since $f^* \geq 0$ with probability 1, it immediately implies the desired lower bound of the lemma

$$\mathbb{E}[f^*] = \Omega\left(\frac{1}{n^{3/2}\log^{1+\alpha} n}\right).$$

This completes the proof of Lemma 10. □

V. Discussion

In this paper, we considered the question of characterizing the PMF for a very general class of networks. For the simpler combinatorial model, we saw that the results for wireless networks follow as simple extensions of those obtained by Leighton and Rao for directed capacitated graphs. For the more interesting information-theoretic setting, we obtain similar characterizations—the extensions are nontrivial in this case and involve the construction of appropriate capacitated graphs to obtain lower and upper bounds on the PMF. The lower bound is constructive, and the simple transmission scheme which achieves it is guaranteed to achieve PMF within an approximation factor which can be obtained in closed form. As a by-product, we obtain scaling laws for geometric random networks for the protocol model and for the information-theoretic case. While most of the results for geometric random networks were known, we provide derivations which are simpler and more systematic.

The characterization of the maximum PMF in terms of the min-cut of appropriate graphs is useful to obtain scaling laws for random networks, and provides intuition as to which properties of the underlying network graphs determine the maximum supportable PMF. However, the computation of the min-cut of a graph is hard. Hence, we provide computational bounds for the maximum PMF as well; these bounds can be computed in polynomial time by solving *linear programs*. We characterize the approximation bound in closed form. One implication is that for the first time, lower bounds are available for a very general class of networks with guaranteed approximation bounds.

Appendix A

Proof: [Lemma 2] From the hypothesis of the lemma, it is clear that for at least $(1-n^{-1-\alpha})$ fraction of all $n!$ permutations in S_n, the permutation flow $(nf)\Sigma$ is feasible. By definition and symmetry of permutations, we can write

$$U_n(f) = \frac{1}{n!}\sum_{i=1}^{n!}(nf)\Sigma_i.$$

Let us define the following indicator function:

$$\mathbf{1}_i = \begin{cases} 1, & (nf)\Sigma_i \text{ is supportable} \\ 0, & \text{otherwise.}\end{cases}$$

Consider a uniform time sharing scheme between all the $n!$ permutation flows. Then the following traffic matrix is supportable:

$$\Gamma_n = \frac{1}{n!}\sum_{i=1}^{n!}\mathbf{1}_i(nf)\Sigma_i.$$

Thus

$$\begin{aligned}||U_n(f) - \Gamma_n|| &= \left|\left|\frac{1}{n!}\sum_{i=1}^{n!}(1-\mathbf{1}_i)(nf)\Sigma_i\right|\right| \\ &\overset{(a)}{\leq} \frac{1}{n!}\sum_{i=1}^{n!}||(1-\mathbf{1}_i)(nf)\Sigma_i|| \\ &\overset{(b)}{=} \frac{1}{n!}\sum_{i=1}^{n!}(1-\mathbf{1}_i)nf \leq \frac{nf}{n!}\frac{n!}{n^{1+\alpha}} \\ &= \frac{f}{n^\alpha} \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Step (a) uses triangle inequality for norms and step (b) uses $||\Sigma_i|| = 1$ for any permutation matrix Σ_i. □

Proof: [Lemma 3] Consider a cut S such that $\pi(S)\pi(S^C) > 0$. Then, the following is a continuous function of C

$$\frac{\sum_{(i,j):i\in S, j\in S^c} C(i,j)}{\pi(S)\pi(S^c)}.$$

The lemma then follows since the minimum of a finite number of continuous functions is continuous. □

Proof: [Lemma 4] For $\hat{C} \in \mathcal{C}$ and any $\epsilon > 0$ define the set

$$\mathcal{B}_\delta = \{C \in \mathcal{C} : \|C - \hat{C}\| < \delta\}.$$

To prove the lemma, we have to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $C \in \mathcal{B}_\delta$, $|f_\pi(C) - f_\pi(\hat{C})| < \epsilon$. For $\hat{C} \in \mathcal{C}$ consider

$$\delta_1 = \min_{(k,m)\in E} \left\{\alpha \hat{C}(k,m) : \hat{C}(k,m) > 0\right\}, \quad 0 < \alpha < 1$$

and

$$\delta = \min\left\{\delta_1, \min_{i,j:\pi(i)\pi(j)>0} \frac{\epsilon}{2\pi(i)\pi(j)}\right\}.$$

Then for any $C \in \mathcal{B}_\delta$, it follows from (1) that $f_\pi(C) \le f_\pi(\hat{C}) + \epsilon$. It only remains to show that $f_\pi(C) \ge f_\pi(\hat{C}) - \epsilon$. For this note that $C \succ \underline{C}$ for all $C \in \mathcal{B}_\delta$, where $\underline{C}$ is as follows:

$$\underline{C}(k,m) = \begin{cases} 0, & C(k,m) = 0 \\ \hat{C}(k,m) - \delta_1, & \text{otherwise.} \end{cases}$$

Now by scaling all the variables by $(1-\alpha)$ in the LP (1) for $\hat{C}$ and using the monotonicity of $f_\pi(C)$ in C, we can see that $f_\pi(C) \ge (1-\alpha) f_\pi(\hat{C})$ for all $C \in \mathcal{B}_\delta$. If $f_\pi(\hat{C}) = 0$, we are done. If not, choose $\alpha = \min(\epsilon/f_\pi(\hat{C}), 0.5)$, which gives $f_\pi(C) \ge f_\pi(\hat{C}) - \epsilon$, and so we are done. □

APPENDIX B

Proof: [Lemma 7] For any $x \in (0,1)$, $x/2 \le \log(1+x) \le x$, so

$$\sum_{i=1}^N \log(1+\sqrt{x_i}) \le \sum_{i=1}^N \sqrt{x_i} \tag{21}$$

$$\le \sqrt{N}\sqrt{\sum_{i=1}^N x_i}$$

$$\le \sqrt{2N}\sqrt{\sum_{i=1}^N \log(1+x_i)} \tag{22}$$

where (22) follows from Cauchy–Schwarz inequality. □

Proof: [Lemma 8] Define

$$f(x) = \frac{1}{\alpha}\log(1+\alpha x) - \log(1+x).$$

Note that $f'(x) \ge 0$ for $x \ge 0$ and $f(0) = 0$. □

Proof: [Lemma 9] We will use the following result, that follows directly from Theorem 1 in [26].

Theorem 8: Consider a complex scalar channel where the output Y when X is transmitted is given by

$$Y = hX + Z + S$$

where Z is a complex circularly symmetric Gaussian random variable with unit variance, and S satisfies $\mathbb{E}[S^*S] \le \hat{P}$. Also, h is zero mean and i.i.d. over channel uses. If X is a complex zero-mean circularly symmetric Gaussian random variable with $\mathbb{E}[X^*X] = P$, then $I(X;(Y,h)) \ge \mathbb{E}\log\left(1 + \frac{P|h|^2}{1+\hat{P}}\right)$.

We consider a transmission scheme where the signal transmitted over each link, when active, is a complex zero-mean white circularly symmetric Gaussian with variance P. Moreover, we assume that the transmissions on all links are mutually independent. Let $t_1, t_2, \ldots$ denote times at which link $(1,2)$ is scheduled. Hence, at any such time $t \in \{t_1, t_2, \ldots\}$, the received signal at node 2 is given by

$$Y_2(t) = H_{21}(t)X_1(t) + \sum_{k\ne 1,2} H_{2k}(t)X_k(t) + Z_2(t).$$

Using the mutual independence of transmissions and zero mean property along with the construction of the scheduling scheme

$$\mathbb{E}\left|\sum_{k\ne 1,2} H_{2k}(t)X_k(t) + Z_2(t)\right|^2 \le 1 + nPg(r(1+\eta)).$$

From Theorem 8

$$I(X_1(t);(Y_2(t),H_{21}(t))) \ge \mathbb{E}\log\left(1 + \frac{P|H_{21}|^2}{(1+nPg(r(1+\eta))}\right). \tag{23}$$

Since the channel is assumed to be i.i.d. over channel uses, a random coding argument can be used to achieve this rate with a probability of error that goes to zero as the block length goes to infinity [12, Ch. 10].

Combining this with the time sharing between different sets of links described in the proof of Theorem 5, since each link gets to transmit at least once in α times slots, or at least $1/\alpha$ fraction of the time, it follows that

$$\lambda_{12} \ge \alpha^{-1}\mathbb{E}\log\left(1 + \frac{P|H_{21}|^2}{1+nPg(r(1+\eta))}\right). \quad □$$

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